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ORIGINAL ARTICLES

Numerical Investigations on Impulsive Fuzzy Differential Equations

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ABSTRACT

In this paper, we proposed a method for computing the approximate solution for impulsive fuzzy differential equations by utilizing the existing results in impulsive differential equations and fuzzy differential equations. The numerical solutions are investigated since many impulsive differential equations cannot be solved analytically or their solving is complicated. The solutions are then compared through the examples.

Key words: Fuzzy Differential Equations, Impulsive Differential Equations, Impulsive Fuzzy Differential Equations, Numerical Methods

Introduction

Differential equations with impulse effect, known as impulsive differential equations are seen as a natural description of observed evolution phenomenon of several real world problems. Therefore, the theory of the impulsive differential equations has been studied extensively by several researchers (Shen, J., 2003; Akhmetov, M.U. and A. Zafer, 2000; Lakshmikantham, V. and F.A. McRae, 2001). Bainov and Kulev in (1988) and (Kulev, G.K. and D.D. Bainov, 1989) investigated the stability and global stability of systems with impulse by the Lyapunov function. Randelovich *et al.*, (2000) gave the numerical algorithm for solving impulsive differential equations. A research on boundary value problems for higher order impulsive differential equations has also been done in (Uğur, Ö. and M.U. Akhmet, 2006). Furthermore, Ran *et al.*, (2008) investigated the numerical methods for impulsive differential equation which is seldom explored by others. The numerical solutions of impulsive differential equations can be found in (Kavikumar, 2010; Kavikumar, 2010).

However, mathematical modeling of real world problems surely encounters indeterminacy resulting by our inability to differentiate events exactly. The main property of indeterminacy is vagueness of its semantics. Classical mathematics could not cope with such vagueness (Lakshmikantham, V. and F.A. McRae, 2001). Therefore, the fuzzy set theory which enables us to describe vagueness is very useful. The theory of fuzzy sets, fuzzy valued functions and necessary calculus of fuzzy function have been investigated in (Dubois, D. and H. Prade, 1982a; Dubois, D. and H. Prade, 1982b; Dubois, D. and H. Prade, 1982c; Klir, G.J., *et al.*, 1997; Goetschel, R.J. and W. Voxman, 1986; Novak, V., 1988). The framework for the study of impulsive fuzzy differential equations has been developed and the basic property of solutions of impulsive fuzzy differential equations is available (Lakshmikantham, V. and F.A. McRae, 2001). However, there are not too many articles on impulsive fuzzy differential equations (Benchohra, M., *et al.*, 2007).

In this paper, the numerical solution of impulsive fuzzy differential equations is done by combining the theories of impulsive differential equations and fuzzy differential equations. The results of investigations are presented through the examples of the first order and the second-order linear fuzzy impulsive differential equations.

Preliminaries:

Let $P_k(\Re^n)$ denote the family of all nonempty compact, convex subsets of $\Re^n$. If $\alpha, \beta \in \Re$ and $A, B \in P_k(\Re^n)$, then $\alpha(A+B) = \alpha A + \alpha B,\ \alpha(\beta A) = (\alpha\beta)A,\ 1A = A$ and

if $\alpha, \beta \ge 0,$ then $(\alpha + \beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0 + a],\ t_0 \ge 0$ and $a > 0$ and denoted by $E^n = [u : \Re^n \to [0,1]$ such that u satisfies (i) and (ii) mentioned below]:

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(i) *u* is normal, that is, there exists an $x_0 \in \Re^n$ such that $u(x_0) = 1$;

(ii) *u* is fuzzy convex, that is, for $x, y \in \Re^n$ and $0 \le \lambda \le 1$,

$$u(\lambda x + (1-\lambda)y) \ge \min[u(x), u(y)];$$

(iii**)** *u* is upper semi continuous;

(iv) $[u]^0 = \overline{[x \in \Re^n : u(x) > 0]}$ is compact.

For $0 < \alpha \le 1$, we denote $[u]^\alpha = [x \in \Re^n : u(x) \ge \alpha]$. Then from (i) to (iv), it follows that the $\alpha-$level sets $[u]^\alpha \in P_k(\Re^n)$ for $0 \le \alpha \le 1$. For later purpose, we define $\hat{o} \in E^n$ as $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \ne 0$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(\Re^n)$. Then we define

$$d[u,v] = \sup_{0 \le \alpha \le 1} d_H\left[[u]^\alpha, [v]^\alpha\right]$$

which defines a metric in E^n and (E^n, d) is a complete metric space.

The following listed the properties of $d[u,v]$:

$d[u+w, v+w] = d[u,v]$ and $d[u,v] = d[v,u]$,

$d[\lambda u, \lambda v] = |\lambda|\ d[u,v]$

$d[u,v] \le d[u,w] + d[w,v]$

for all $u, v, w \in E^n$ and $\lambda \in R$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the $H-$difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

Moreover, if $F : I \to E^n$ is continuous, then it is integrable and

$$\int_a^b F = \int_a^c F + \int_c^b F$$

Also, the following properties of the integral are valid. If $F, G : I \to E^n$ can be integrated, $\lambda \in R$, then the following hold:

$\int (F + G) = \int F + \int G;$

$\int \lambda F = \lambda \int F, \qquad \lambda \in R;$

$d[F, G]$ is integrable

$d\left[\int F, \int G\right] \le \int d[F, G]$

Finally, let $F : I \to E^n$ be continuous. Then the integral $G(t) = \int_{t_0}^{t} F$ is differentiable and $G'(t) = F(t)$. Furthermore, $F(t) - F(t_0) = \int_\alpha^t F'(t)$

Consider the fuzzy differential equation system

$$u'(t) = f(t,u), \qquad u(t_0) = u_0 \tag{2.1}$$

where $f \in C[I \times E^n, E^n]$ and $I = [t_0, t_0 + a], t_0 \geq 0, a > 0.$ Before proceeding any further, we note that a mapping $u : I \to E^n$ is a solution of the initial value problem (2.1) if and only if it is continuous and satisfies the integral equation,

$$u(t) = u_0 + \int_0^t f(s, u(s))ds \text{ for } t \in I$$

An application of contraction mapping principle yields the existence and uniqueness result.

Theorem 2.1 (Kaleva, O., 1987):

Assume that $f \in C[I \times E^n, E^n]$ and satisfies $d[f(t,u), f(t,v)] \leq Ld[u,v],\ L > 0,$ for $(t,u),(t,v) \in I \times E^n.$ Then, the initial value problem (2.1) has a unique solution $u(t) = u(t, t_0, u_0)$ on *I.*

Theorem 2.2 (Kaleva, O., 1987)

Let $F : T \to E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)], \alpha \in [0,1].$ Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].$

Definition 2.1 (Lakshmikantham, V., et al., 1989):

A system of differential equation of the form

$$\frac{dx}{dt} = f(t, x), \quad t \neq t_k \tag{2.2}$$

with conditions

$$\Delta x|_{t=t_k} = x(t_k{}^+) - x(t_k{}^-) = I_k(x(t_k))$$

where $I_k : \Re^n \to \Re^n$ are continuous operators, $k = 0, \pm 1, \pm 2, ...$ is called impulsive differential equation at fixed impulse.

Definition 2.2 (Bainov, D.D. and P.S. Simenov, 1989):

The function $x : \Re_+ \to \Re$ is said to be a solution of the system (2.2) if the following conditions are satisfied :

(1) $\lim_{t \to 0^+} x(t) = x_0 = x(0^+)$.

(2) For $t \in (0, \infty), t \neq t_k$, the function $x(t)$ is differentiable and $x'(t) = f(t, x)$.

(3) The function $x(t)$ is left continuous in $(0, \infty),$ and if $t \in (0, \infty), t = \tau_k$ then $x(t^+) = x(t^-) + I_k(x(t_k))$.

Definition 2.3 (Buckley, J.J. and T. Feuring, 2000):

The Seikkala derivative of $x(t)$ is defined as follows:

If $[x'_1(t, \alpha), x'_2(t, \alpha)]$ are the α-cuts of a fuzzy number for each $t \in I,$ then $x(t)$ exists and $x(t)[\alpha] = [x'_1(t, \alpha), x'_2(t, \alpha)].$

Methodology:

One of the main topics of this research is the development of the algorithm in solving impulsive differential equation as well as the impulsive fuzzy differential equation. Analytical results on stability properties and comparison theorems are referred to (Vatsala, A.S., 2003; Lakshmikantham, V. and R. Mohapatra, 2003; Lakshmikantham, V. and X.Z. Liu, 1999). The study of impulsive fuzzy differential equations has been done by utilizing the existing results in impulsive and fuzzy differential equations, and also by considering the fuzzy differential equations with the numerical methods.

The numerical solutions of the purposed union are based on the Seikkala derivative. According to Buckley and Feuring (Buckley, J.J. and T. Feuring, 2000), the Seikkala solution is the most general solution to the fuzzy initial value problems. If these solutions are defined by the α-level cuts of a triangular fuzzy number, then the solutions of impulsive fuzzy differential equations is expected to exist, which has been concluded from the numerical values.

The procedures are going to be done as follow :

Development of Fuzzy Differential Equation:

Consider the fuzzy differential equation

$$y' = f(t, y, k),\ y(0) = c\,.$$

The equation has a unique solution

$y = g(t,k,c)$, for $t \in I, k \in K \subset R^n, c \in C \subset R^n$.

The value of the k_i and c are uncertain and model this uncertainty by substituting triangular fuzzy numbers. Now solve for y which will be a fuzzy function. This new solution for fuzzy y is the topic of this research.

Stability Criteria:

Stability property by comparison result in terms of a Lyapunov function is very important. It can be proved via the theory of differential inequalities. Here, the Lyapunov function serves as a vehicle to transform the fuzzy differential equation into a scalar comparison differential equations (Lakshmikantham, V. and R. Mohapatra, 2003). Therefore, it is enough to consider the stability properties of the simpler comparison equation.

Procedure for the numerical method:

The solution can be calculated using the α -level cut method by the application of the Euler, Second-order Taylor Series, Modified Euler and Runge-Kutta fourth-order method using a representation of fuzzy numbers and studies the impulsive initial value problem.

The structure of the algorithm :

(i) Calculate α -level cut of fuzzy parameters.
(ii) Calculate the solution of impulsive fuzzy differential equations with interval parameters.
(iii) Calculate the fuzzy membership function of the solution.

Instrument:

The numerical computation is done by using the MATLAB mathematical software. MATLAB is chosen because of its outstanding performance in solving mathematical problems, especially fuzzy differential equations (Gilat, A., 2005; Sivanandam, S.N., *et al*., 2007). Furthermore, MATLAB offers a very user friendly interface with extensive visualization and numerical computation capabilities (Lynch, S., 2004).

Numerical Solutions of Impulsive Fuzzy Linear Differential Equations:

Consider the impulsive fuzzy differential equation

$$\begin{cases} u'(t) = f(t,u), & t \neq t_k \\ I_k(u(t_k)) = u(t_k^+) - u(t_k^-), & t = t_k \\ u(t_0) = u_0 \end{cases} \tag{4.1}$$

where $f : \Re_+ \times E^n \to E^n$, f is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n$, $\lim f(t,v) = f(t_k^+, u)$ exists as $(t,v) \to (t_k^+, u)$. Also $I_k : E^n \to E^n$ and $u_0 \in E^n$. If the assumptions of Theorem 2.1 hold on each set $[t_{k-1}, t_k] \times E^n$, then clearly there exists a unique solution

$u_i(t) = u(t, t_i, u_{i-1}(t_i^+))$ on each interval $[t_{i-1}, t_i]$. As a result, employing the impulsive condition in Equation (4.1) at each $t = t_i$, we can define the solution $u(t)$ of Equation (4.1) on the interval $[t_0, \infty)$.

By using the representation of fuzzy numbers studied by Goetschel and Voxman (1986) and Wu and Ma (1991), $u \in E^n$ is presented by a pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \le \alpha \le 1$, such that (i) $\underline{u}(\alpha)$ is bounded, left continuous, and non-decreasing, (ii) $\overline{u}(\alpha)$ is bounded, left continuous, and non-increasing, and (iii) $\underline{u}(\alpha) \le \overline{u}(\alpha)$, $0 \le \alpha \le 1$.

By applying Theorem 2.2, Equation (2.1) can be replaced by an equivalent system:

$$\begin{cases} \underline{u}'(t) = \underline{f}(t,u) = F_k(t, \underline{x}, \overline{x}), & \underline{I}_k(\underline{u}(t_k)) = \underline{u}(t_k^+) - \underline{u}(t_k^-), \quad \underline{u}(t_0) = \underline{u}_0 \\ \overline{u}'(t) = \overline{f}(t,u) = G_k(t, \underline{x}, \overline{x}), & \overline{I}_k(\overline{u}(t_k)) = \overline{u}(t_k^+) - \overline{u}(t_k^-), \quad \overline{u}(t_0) = \underline{u}_0 \end{cases} \tag{4.2}$$

which possesses a unique solution $(\underline{u}, \overline{u})$ which is a fuzzy function. That is for each t, the pair $[\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$ is a fuzzy number, where $[\underline{u}(t;\alpha), \overline{u}(t;\alpha)]$ are respectively the solutions of the parametric form given by:

$$\begin{cases} \underline{u}'(t;\alpha) = F_k(t, \underline{x}(t;\alpha), \overline{x}(t;\alpha)), & \underline{I}_k(\underline{u}(t_k;\alpha)) = \underline{u}(t_k^+;\alpha) - \underline{u}(t_k^-;\alpha), \quad \underline{u}(t_0;\alpha) = \underline{u}_0(\alpha) \\ \overline{u}'(t;\alpha) = G_k(t, \underline{x}(t;\alpha), \overline{x}(t;\alpha)), & \overline{I}_k(\overline{u}(t_k;\alpha)) = \overline{u}(t_k^+;\alpha) - \overline{u}(t_k^-;\alpha), \quad \overline{u}(t_0;\alpha) = \underline{u}_0(\alpha) \end{cases}$$

for $\alpha \in [0,1]$. (4.3)

For a fixed α, to integrate the system in $[t_0, t_1], (t_1, t_2], \ldots, (t_k, t_{k+1}], \ldots$, we replace each interval by a set of $N_k + 1$ discrete equally spaced grid points (including the endpoints) at which the exact solution $(\underline{u}(t;\alpha), \overline{u}(t;\alpha))$ is approximated by some $(\underline{y_k}(t;\alpha), \overline{y_k}(t;\alpha))$. For the chosen grid points on $(t_k, t_{k+1}]$ at $t_{k,n} = t_{k,0} + nh_k,\ h_k = \dfrac{t_{k+1} - t_k}{N_k}$, $0 \le n \le N_k$, let $(\underline{Y_k}(t;\alpha), \overline{Y_k}(t;\alpha)) \equiv (\underline{u_k}(t;\alpha), \overline{u_k}(t;\alpha))$. $(\underline{Y_k}(t;\alpha), \overline{Y_k}(t;\alpha))$ and $(\underline{y_k}(t;\alpha), \overline{y_k}(t;\alpha))$ may be denoted respectively by $(\underline{Y_{k,n}}(\alpha), \overline{Y_{k,n}}(\alpha))$ and $(\underline{y_k}(t;\alpha), \overline{y_k}(t;\alpha))$. We allow the N_k's to vary over the $(t_k, t_{k+1}]$'s so that the h_k's may be comparable.

The Euler method is the first order approximation of $\underline{Y_k'}(t;\alpha)$ and $(\overline{Y_k'}(t;\alpha)$, which can be written as:

$$\begin{cases} \underline{Y}_{k,n+1}(\alpha) \approx \underline{Y}_{k,n}(\alpha) + h_k F_k[t_{k,n}, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)], \\ \overline{Y}_{k,n+1}(\alpha) \approx \overline{Y}_{k,n}(\alpha) + h_k G_k[t_{k,n}, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)]. \end{cases} \tag{4.4}$$

Then by Equation (4.4) ,

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) \approx \underline{y}_{k,n}(\alpha) + h_k F_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)], \\ \overline{y}_{k,n+1}(\alpha) \approx \overline{y}_{k,n}(\alpha) + h_k G_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)]. \end{cases} \tag{4.5}$$

However, Equation (4.5) will use $\underline{y}_{0,0}(\alpha) = \underline{u}_0(\alpha)$, $\overline{y}_{0,0}(\alpha) = \overline{u_0}(\alpha)$ and $\underline{y}_{k,0}(r) = \underline{y}_{k-1,N_{k-1}}(r)$, $\overline{y}_{k,0}(\alpha) = \overline{y}_{k-1,N_{k-1}}(\alpha)$ if $k \ge 1$. Then Equation (4.5) represents an approximation of $\underline{Y}_k(t;\alpha)$ and $(\overline{Y_k}(t;\alpha)$ for each of the intervals $t_0 \le t \le t_1,\ t_1 < t \le t_2, \ldots, t_k < t \le t_{k+1}, \ldots$

Similarly, any other numerical methods can be written the same way as Equation (4.4) and Equation (4.5) according to the method's formula.

The Second-order Taylor Series method can be written as :

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) \approx \underline{y}_{k,n}(\alpha) + h_k F_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] + \dfrac{(h_k)^2}{2!}\underline{y}''_{k,n}(\underline{\xi}_n) \\ \overline{y}_{k,n+1}(\alpha) \approx \overline{y}_{k,n}(\alpha) + h_k G_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] + \dfrac{(h_k)^2}{2!}\overline{y}''_{k,n}(\overline{\xi}_n)] \end{cases} \tag{4.6}$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$.

The Modified Euler method is given by :

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) \approx \underline{y}_{k,n}(\alpha) + \dfrac{h_k}{2}(F_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] + F_k[t_{k,n+1}, \underline{y}_{k,n+1}(\alpha), \overline{y}_{k,n+1}(\alpha)]) \\ \overline{y}_{k,n+1}(\alpha) \approx \overline{y}_{k,n}(\alpha) + \dfrac{h_k}{2}(G_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] + G_k[t_{k,n+1}, \underline{y}_{k,n+1}(\alpha), \overline{y}_{k,n+1}(\alpha)]) \end{cases} \tag{4.7}$$

The Runge-Kutta fourth-order method can be written as :

$$\underline{y}_{k,n+1}(\alpha) \approx \underline{y}_{k,n}(\alpha) + \frac{1}{6}\left[\underline{K}_{k,1} + 2\underline{K}_{k,2} + 2\underline{K}_{k,3} + \underline{K}_{k,4}\right]$$

$$\underline{K}_{k,1} = hF_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)]$$

$$\underline{K}_{k,2} = hF_k\left[t_{k,n} + \frac{h}{2}, \underline{y}_{k,n}(\alpha) + \frac{1}{2}\underline{K}_{k,1}, \overline{y}_{k,n}(\alpha) + \frac{1}{2}\underline{K}_{k,1}\right]$$

$$\underline{K}_{k,3} = hF_k\left[t_{k,n} + \frac{h}{2}, \underline{y}_{k,n}(\alpha) + \frac{1}{2}\underline{K}_{k,2}, \overline{y}_{k,n}(\alpha) + \frac{1}{2}\underline{K}_{k,2}\right]$$

$$\underline{K}_{k,4} = F_k[t_{k,n} + h, \underline{y}_{k,n}(\alpha) + \underline{K}_{k,3}, \overline{y}_{k,n}(\alpha) + \underline{K}_{k,3}]$$

and for

$$\overline{y}_{k,n+1}(\alpha) \approx \overline{y}_{k,n}(\alpha) + \frac{1}{6}\left[\overline{K}_{k,1} + 2\overline{K}_{k,2} + 2\overline{K}_{k,3} + \overline{K}_{k,4}\right]$$

$$\overline{K}_{k,1} = hG_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)]$$

$$\overline{K}_{k,2} = hG_k\left[t_{k,n} + \frac{h}{2}, \underline{y}_{k,n}(\alpha) + \frac{1}{2}\overline{K}_{k,1}, \overline{y}_{k,n}(\alpha) + \frac{1}{2}\overline{K}_{k,1}\right]$$

$$\overline{K}_{k,3} = hG_k\left[t_{k,n} + \frac{h}{2}, \underline{y}_{k,n}(\alpha) + \frac{1}{2}\overline{K}_{k,2}, \overline{y}_{k,n}(\alpha) + \frac{1}{2}\overline{K}_{k,2}\right]$$

$$\overline{K}_{k,4} = hG_k[t_{k,n} + h, \underline{y}_{k,n}(\alpha) + \overline{K}_{k,3}, \overline{y}_{k,n}(\alpha) + \overline{K}_{k,3}]$$

Therefore,

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) \approx \underline{y}_{k,n}(\alpha) + \dfrac{1}{6}\left[\underline{K}_{k,1} + 2\underline{K}_{k,2} + 2\underline{K}_{k,3} + \underline{K}_{k,4}\right] \\ \overline{y}_{k,n+1}(\alpha) \approx \overline{y}_{k,n}(\alpha) + \dfrac{1}{6}\left[\overline{K}_{k,1} + 2\overline{K}_{k,2} + 2\overline{K}_{k,3} + \overline{K}_{k,4}\right] \end{cases} \tag{4.8}$$

Numerical Examples:

The following examples illustrate the results of this chapter. Examples 4.1 and 4.2 are taken from (Randelovich, B.M., *et al.*, 2000). However, in (Randelovich, B.M., *et al.*, 2000) the examples are not solved under the fuzzy environment.

Example 4.1:

Consider the first order linear impulsive fuzzy differential equations

$$\frac{dx}{dt} = f(t, x), \quad t \neq t_k$$
$$\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, ... \tag{4.9}$$
$$x_0 = x(t_0)$$

and $t_0 = 0.0,$ $\quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} \alpha - 2, & -\alpha \\ \alpha - 1, & 1 - \alpha \end{bmatrix}$

$$f(t, x) = \begin{bmatrix} 0.1666666x_1 + 0.1666666x_2 + 0.1666666 \\ -0.1666666x_1 - 0.1666666x_2 + 0.5833333 \end{bmatrix}$$

The impulsive operators act at $t_1 = 1.0$ and $t_2 = 2.0$ are given as follow.

$$I_1 = \begin{bmatrix} 0.25 & 0.25 \\ 0.0 & -1.0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 3.0 & 4.0 \\ 0.0 & -1.0 \end{bmatrix} \tag{4.10}$$

Here, we wish to investigate the numerical solutions after the jump points at *t* = 1.0 and *t* = 2.0. The analytical solution at $\alpha = 1$ is given by

$$x = \begin{cases} x_1(t) = 0.0625t^2 - 1.0, & t \in (-\infty, 1) \\ x_2(t) = -0.0625t^2 + 0.75t, & t \in (-\infty, 1) \\ \\ x_1(t) = 0.0625t^2 - 1.0625, & t \in [1, 2) \\ x_2(t) = -0.0625t^2 + 0.75t - 0.6875, & t \in [1, 2) \\ \\ x_1(t) = 0.0625t^2 - 1.25, & t \in [2, \infty) \\ x_2(t) = -0.0625t^2 + 0.75t - 1.25, & t \in [2, \infty) \end{cases} \tag{4.11}$$

A comparison between the analytical and the approximate solutions at *t* = 1.0 and *t* = 2.0 after the jump points are shown in Table 4.1, Table 4.2, Table 4.3 and Table 4.4.

In Table 4.1, the approximate solutions of $\underline{x}_1$ at $t = 1.0^+$ given by the Modified Euler method has no errors at all when being compared to the Euler method. This is true for all levels of α.

Table 4.1: The numerical solutions of $\underline{x}_1$ at $t = 1.0^+$

α	Euler	Modified Euler	Analytical	Error Euler	Error Modified Euler
0	-2.8396	-2.8333	-2.8333	0.0063	0.0000
0.2	-2.4729	-2.4667	-2.4667	0.0062	0.0000
0.4	-2.1062	-2.1000	-2.1000	0.0062	0.0000
0.6	-1.7396	-1.7333	-1.7333	0.0063	0.0000
0.8	-1.3729	-1.3667	-1.3667	0.0062	0.0000
1.0	-1.0063	-1.0000	-1.0000	0.0063	0.0000

Table 4.2 gives the approximate solutions of $\underline{x}_1$ at $t = 2.0^+$ between the Modified Euler method and the Euler method. The Modified Euler shows better accuracy in its solutions than the Euler method if being compared to the analytical solutions.

Table 4.2: The numerical solutions of $\underline{x}_1$ at $t = 2.0^+$

α	Euler	Modified Euler	Analytical	Error Euler	Error Modified Euler
0	-8.3583	-8.3333	-8.3332	0.0251	0.0001
0.2	-6.8917	-6.8667	-6.8668	0.0249	0.0001
0.4	-5.4250	-5.4000	-5.4000	0.0250	0.0000

0.6	-3.9583	-3.9333	-3.9332	0.0251	0.0001
0.8	-2.4917	-2.4667	-2.4668	0.0249	0.0001
1.0	-1.0250	-1.000	-1.0000	0.0250	0.0000

In Table 4.3, the approximate solutions given by the Modified Euler of $\overline{x}_1$ at $t = 1.0^+$ shows the same result as Table 4.1 with no errors at all level sets. It shows that the Modified Euler has better accuracy for the approximate values of $\overline{x}_1$ at $t = 1.0^+$.

Table 4.3: The numerical solutions of $\overline{x}_1$ at $t = 1.0^+$

α	Euler	Modified Euler	Analytical	Error Euler	Error Modified Euler
0	0.8271	0.8333	0.8333	0.0062	0.0000
0.2	0.4604	0.4667	0.4667	0.0063	0.0000
0.4	0.0937	0.1000	0.1000	0.0063	0.0000
0.6	-0.2729	-0.2667	-0.2667	0.0062	0.0000
0.8	-0.6396	-0.6333	-0.6333	0.0063	0.0000
1.0	-1.0063	-1.0000	-1.0000	0.0063	0.0000

Table 4.4 also shows that the Modified Euler has less error than the Euler method in its solutions at all level sets when being compared to the analytical solutions.

Table 4.4: The numerical solutions of $\overline{x}_1$ at $t = 2.0^+$

α	Euler	Modified Euler	Analytical	Error Euler	Error Modified Euler
0	6.3083	6.3333	6.3332	0.0249	0.0001
0.2	4.8417	4.8667	4.8668	0.0251	0.0001
0.4	3.3750	3.4000	3.4000	0.0250	0.0000
0.6	1.9083	1.9333	1.9332	0.0249	0.0001
0.8	0.4417	0.4667	0.4668	0.0251	0.0001
1.0	-1.0250	-1.0000	-1.0000	0.0250	0.0000

It is very interesting to see and validate the fact that better numerical method influences the accuracy of the solutions. It can be clearly seen that it is possible to have the numerical solutions of impulsive fuzzy differential equations. However, the solutions of $\underline{x}_2$ and $\overline{x}_2$ are the same values as the exact solutions. This is mainly due to the impulsive operators that act at *t* = 1.0 and *t* = 2.0 given in (4.10). Therefore, the solutions are not compared in the tables.

Example 4.2:

Consider a ball that is jumping on a flat horizontal surface. Let the loss of energy which is caused by friction of the surface is defined by a constant, μ.

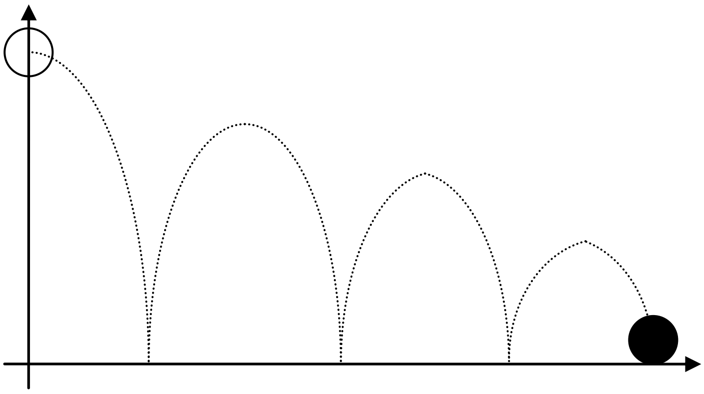

Fig. 4.1: A jumping ball

This process can be described as the differential equation of second order,

$$m\frac{d^2x}{dt^2} = F \tag{4.12}$$

where *m* is the mass of the ball, *F* = *-mg*, is the force ($g \approx 9.81\frac{m}{s^2}$ is acceleration of the earth gravitation). At times when the ball touches the surface, the vertical component of the vector of velocity changes its sign. The state of this process is described by the vertical position, the velocity and the horizontal position of the ball. Then, the approximated values of the three coordinates can be sought at the moment of interest. This process can be described with the impulsive differential equations

$$\frac{dx_1}{dt} = x, \quad \frac{dx_2}{dt} = -g, \quad \frac{dx_3}{dt} = v_0 \tag{4.13}$$

with the initial condition,

$$x = (x_1, x_2, x_3) = (h_0, 0, 0), \;\; h_0 = (0.1 + 0.1\alpha \,, \;\; 0.3 - 0.1\alpha)\,m, \;\; v_0 = 1m/s.$$

and the condition of jump is

$$I_k(x_1, x_2, x_3) = (x_1, -\mu \cdot x_2, x_3) \text{ for } \mu(x) = x_1 = 0. \tag{4.14}$$

Assume μ = 0.91, where x_1 represents the vertical position, x_2 is the velocity, and x_3 is the horizontal position of the ball. Let the interest moment to be approximated is at *t* = 0.5. The step size is fixed to *h* = 0.00001.

The analytical solution of Example 4.2 at $\alpha = 1$ is given by

$$\begin{aligned} x_1(t) &= -4.905t^2 + 0.2 \\ x_2(t) &= -9.81t \qquad\qquad , \quad t \in [0.0000,\, 0.2019] \\ x_3(t) &= t \\ x_1(t) &= -4.905t^2 + 3.7835t - 0.5639 \\ x_2(t) &= -9.81t + 3.7835 \qquad\qquad , \quad t \in [0.2019,\, 0.5694] \\ x_3(t) &= t \end{aligned} \tag{4.15}$$

A comparison between the analytical and the approximate solutions at *t* = 0.5 with different step size by using the Euler method are shown in Table 4.5, Table 4.6, Figure 4.2 and Figure 4.3.

It can be seen from Table 4.5 and Table 4.6 that the small step size gives better accuracy. The step size of *h* = 0.001 has less error than *h* = 0.01. The error for each level is given as the maximum error between the lower and the upper values.

Table 4.5: The numerical solutions of ($\underline{x}_1$, $\overline{x}_1$) at $t = 0.5$ between *h* = 0.01 and*h* = 0.001

α	Euler (*h* = 0.01) (A)	Euler (*h* = 0.001) (B)	Analytical	Error (A)	Error (B)
0	0.0720, 0.2697	0.0676, 0.2472	0.0664, 0.2447	0.0250	0.0025
0.2	0.0447, 0.2497	0.0571, 0.2451	0.0578, 0.2225	0.0272	0.0226
0.4	0.0020, 0.2258	0.0274, 0.2003	0.0296, 0.1974	0.0284	0.0029
0.6	0.0527, 0.1994	0.0183, 0.1723	0.0146, 0.1693	0.0301	0.0030
0.8	0.0968, 0.1694	0.0644, 0.1406	0.0608, 0.1375	0.0319	0.0031
1.0	0.1353, 0.1353	0.1048, 0.1048	0.1016, 0.1016	0.0337	0.0032

Table 4.6: The numerical solutions of ($\underline{x}_2$, $\overline{x}_2$) at $t = 0.5$ between *h* = 0.01 and *h* = 0.001

α	Euler (*h* = 0.01) (A)	Euler (*h* = 0.001) (B)	Analytical	Error (A)	Error (B)
0	0.5706, -0.1758	0.2407, -0.2620	0.2049, -0.2711	0.3657	0.0358
0.2	1.0624, -0.3332	0.7292, -0.3428	0.6927, -0.4283	0.3697	0.0855
0.4	-1.6423,- 0.4943	1.1762, -0.5821	1.1412, -0.5911	2.7835	0.0350
0.6	-1.4229, -0.6629	-1.5116, -0.7510	-1.5209, -0.7603	0.0980	0.0093
0.8	-1.2175, -0.8390	-1.3061, -0.9274	-1.3156, -0.9368	0.0981	0.0095
1.0	-1.0245, -1.0245	-1.1126, -1.1126	-1.1215, -1.1215	0.0970	0.0089

In Figure 4.2, the step size used for the Euler method is *h* = 0.01. If the result is compared to the graph given in Figure 4.3, it shows that the step size *h* = 0.001 gives better accuracy to the analytical solution.

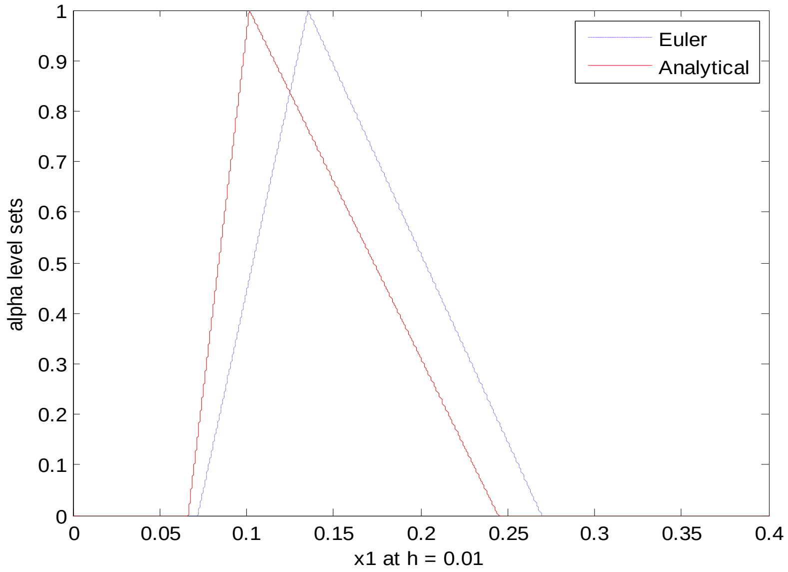

Fig. 4.2: The approximate values of x_1 and *h* = 0.01

In Figure 4.3, the values of x_1 at *t* = 0.5 between the Euler method and the Analytical method shows better accuracy. The step size used is at *h* = 0.001.

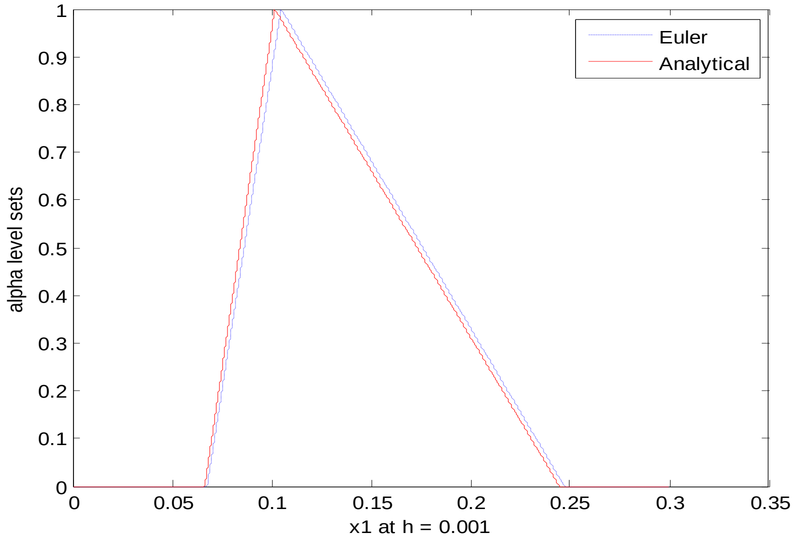

Fig. 4.3: The approximate values of x_1 and *h* = 0.001

Table 4.5 and Table 4.6 have shown that the step size taken has influenced the size of errors. Here, the solutions of x_3 are not being compared because $x_3 = t$ and the algorithm has been programmed until *t* = 0.5 to obtain accurate approximation at the particular moment. The approximate solutions of x_1 are only compared and graphed between the different level sets of fuzzy numbers because the fuzzy values is only concern for x_1. Due to the nature of problem, the initial values cannot be changed for x_2 and x_3. Therefore, it is impossible to graph the level sets of fuzzy numbers for x_2 and x_3.

A comparison of results between several numerical methods other than the Euler method at *t* = 0.5 for Example 4.2 are given in Table 4.7 and Table 4.8. The solutions of the Euler method can easily be referred to Table 4.5 and Table 4.6. The step size used is *h* = 0.01 for all methods. Again, the solutions of x_3 is not compared for the same reason given above.

Table 4.7 shows the solutions of ($\underline{x}_1, \overline{x}_1$) at $t = 0.5$ between the Second-order Taylor Series method, the Modified Euler method, the Runge-Kutta of fourth-order method and the Analytical method. The numerical solutions are compared to the analytical solutions in order to ensure convergence.

Table 4.7: The numerical solutions of ($\underline{x}_1, \overline{x}_1$) at $t = 0.5$

α	Taylor	Modified Euler	Runge-Kutta	Analytical
0	0.0664, 0.2447	0.0664, 0.2447	0.0664, 0.2447	0.0664, 0.2447
0.2	0.0579, 0.2225	0.0578, 0.2225	0.0578, 0.2225	0.0578, 0.2225
0.4	0.0297, 0.1975	0.0297, 0.1975	0.0297, 0.1975	0.0296, 0.1974
0.6	0.0146, 0.1693	0.0146, 0.1693	0.0146, 0.1693	0.0146, 0.1693
0.8	0.0607, 0.1374	0.0609, 0.1375	0.0609, 0.1375	0.0608, 0.1375
1.0	0.1014, 0.1014	0.1015, 0.1015	0.1015, 0.1015	0.1016, 0.1016

Table 4.8 presents the solutions of ($\underline{x}_2$, $\overline{x}_2$) at $t = 0.5$ between the Second-order Taylor Series method, the Modified Euler method, the Runge-Kutta method and the analytical method. Each method gives the lower and upper values of its solutions.

Table 4.8: The numerical solutions of ($\underline{x}_2, \overline{x}_2$) at $t = 0.5$

α	Taylor	Modified Euler	Runge-Kutta	Analytical
0	0.2041,-0.2713	0.2046,-0.2713	0.2054,-0.2713	0.2049, -0.2711
0.2	0.6921,-0.4283	0.6927,-0.4285	0.6921,-0.4283	0.6927, -0.4283
0.4	1.1400,-0.5911	1.1408,-0.5914	1.1403,-0.5911	1.1412, -0.5911
0.6	-1.5211,-0.7603	-1.5211,-0.7603	-1.5211,-0.7603	-1.5209, -0.7603
0.8	-1.3161,-0.9372	-1.3153,-0.9366	-1.3156,-0.9368	-1.3156, -0.9368
1.0	-1.1218,-1.218	-1.1217,-1.1217	-1.1215,-1.1215	-1.1215, -1.1215

Table 4.9 shows the errors for ($\underline{x}_1, \overline{x}_1$) at $t = 0.5$ between the Euler method, The Second-order Taylor Series method, the Modified Euler method and the Runge-Kutta method. The solutions of the Euler method are referred to Table 4.5. The errors of the numerical solutions are compared to the analytical solutions. The value of errors given by the Euler is greater than the other numerical methods. Meanwhile, the value of errors for the Second-order Taylor Series method and the Modified Euler method complements to the errors offered by the Runge-Kutta method.

Table 4.9: The value of errors of ($\underline{x}_1, \overline{x}_1$) at $t = 0.5$ between methods

α	Error Euler	Error Taylor	Error Modified Euler	Error Runge-Kutta
0	0.0056, 0.0250	0.0000, 0.0000	0.0000, 0.0000	0.0000, 0.0000
0.2	0.0131, 0.0272	0.0001, 0.0000	0.0000, 0.0000	0.0000, 0.0000
0.4	0.0276, 0.0284	0.0001, 0.0001	0.0001, 0.0001	0.0001, 0.0001
0.6	0.0381, 0.0301	0.0000, 0.0000	0.0000, 0.0000	0.0000, 0.0000
0.8	0.0360, 0.0319	0.0001, 0.0001	0.0001, 0.0000	0.0001, 0.0000
1.0	0.0337, 0.0337	0.0002, 0.0002	0.0001, 0.0001	0.0001, 0.0001

Table 4.10 shows the errors for ($\underline{x}_2, \overline{x}_2$) at $t = 0.5$ between the Euler method, the Second-order Taylor Series method, the Modified Euler method and the Runge-Kutta method. The solutions of the Euler method are referred to Table 4.6. It shows that better accuracy with less error is given by the Second-order Taylor Series, Modified Euler and the Runge-Kutta method. As for x_2, the Runge-Kutta method shows accurate values of solutions for the upper and lower values when the level sets are $\alpha \geq 0.8.$ The Runge-Kutta method also offers accurate approximations for the upper levels for $\alpha \geq 0.2$ with no errors at all.

Table 4.10: The value of errors of ($\underline{x}_2, \overline{x}_2$) at $t = 0.5$ between methods

α	Error Euler	Error Taylor	Error Modified Euler	Error Runge-Kutta
0	0.3657, 0.0953	0.0008, 0.0002	0.0003, 0.0002	0.0003, 0.0002
0.2	0.3697, 0.0951	0.0006, 0.0000	0.0000, 0.0002	0.0006, 0.0000
0.4	2.7835, 0.0968	0.0012, 0.0000	0.0005, 0.0003	0.0009, 0.0000
0.6	0.0980, 0.0974	0.0002, 0.0000	0.0002, 0.0000	0.0002, 0.0000
0.8	0.0981, 0.0978	0.0003, 0.0004	0.0003, 0.0002	0.0000, 0.0000
1.0	0.0970, 0.0970	0.0003, 0.0003	0.0002, 0.0002	0.0000, 0.0000

Conclusion:

The accuracy of the results for impulsive fuzzy differential equations can be improved by investigating the solutions of the other numerical methods. In this paper, a general numerical procedure for treating the impulsive fuzzy differential equations at fixed moments is proposed. The solutions of using different step size for the Euler method and the solutions between the Euler, Taylor, Modified Euler and Runge-Kutta methods are discussed.

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