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ORIGINAL ARTICLE

Application of the Fixed Point Iterative Procedure to Newton's Method

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ABSTRACT

Let F be an operator mapping a set X into itself. A point $x \in X$ is called a fixed point of F if $x = F(x)$. Hence finding a fixed point on an operator F is equivalent to obtaining a solution of $f(x) = 0$ By this research work, we consider the contraction mapping principle and its application in the solution of the non linear integral equation of radiative transfer.

Key words: Fixed Point, Newton's method, Convergence, Fredholm nonlinear integral equation.

Introduction

We now consider the Newton's method of solution for integral equations, first, we consider the linearization of equations.

Linearization of Equations:

Let F be a Fretchet differentiable operator mapping a subset of a Banach space X into a Banach space Y.
Consider the equation

$$F(x)= 0 \tag{1.1}$$

The principle method for constructing successive approximations x_n to the solution x* of (1.1) is based on successive linearization of the equation. If the approximation to x_n exists, then x_{n+1}, can be computed by replacing

(1.1) by $F^{/}(x_n) + F^{/}(x_n)(x_{n+1}! - x_n) = 0$ (1.2)

if $[F^{/}(x_n)]^{-1} \in L(Y,X)$, then approximation $X_n + 1$ is given by $x_{n+1} = x_n - [F^{/}(x_n)]^{-1} F^{/}(x_n)$, $n \geq 0$ (1.3)

The iteration procedure generated by (1.3) is known as Newton- kantrovich method.

Convergence of Newton's Method:

We seek the condition for which the iteration sequence $\{X_n\}$ defined by (1.3) will converge to a solution $x = x^*$ of the nonlinear equation.
$F(x) = 0$

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Define the operator G by

$G(x) = x - (F^{/}(x))^{-1} F(x)$ (2.1)

Then, the Newton – kantrovich method may be regarded as the usual iteration method,

$X_{n+1} = G(x_n),\ n \geq 0$ (2.2)

For approximating the solution x* of the equation

$x = G(x)$ (2.3)

Suppose that

$\lim_{n \to \infty} x_n = x^*,$ (2.4)

we investigate under what conditions of F and F' the point x^* is a solution of (1.1)

Proposition 1 Rall (1969):

If F is continuous at $x = x^*$, then we have

$F(x^*) = 0$ (2.5)

Proof:

The approximation x_n, satisfies the equation

$F(x_n)\ (x_{n+1} - x_n) = F(x_n)$ (2.6)

Since the continuity of F at x^* follows from the continuity of F´, and taking the limit as $n \to \infty$ in (2.6), (2.5) is obtained.

Proposition 2 Argyrols (2005):

If $\| F'(x) \| \leq b$ (2.7)

in some closed ball which contains $\{x_n\}$, then x^* is a solution of F(x) = 0.

Proof:

As $x_n \to x^*$ we arrive at the result.

$\lim_{n \to \infty} F(x_n) = F(x^*)$ (2.8)

And since
$\|F(x_n)\| \leq b\|x_{n+1} - x_n\|$ (2.9)

(2.5) is obtained by taking the limit as $n \to \infty$ in (2.9)

Proposition 3 Argyros (2005):

If $\| F'(x) \| \leq k$…2.10

In some closed ball $B(x_0, r)$, $0 <r> \infty$ which contains $\{x_n\}$, then x* is a solution of the equation (1.1).

Proof :

$$\|F^{/}(x) - F^{/}(x_0)\| \leq k \| x- x_0\| \leq k r \quad (2.11)$$

So the condition of proposition 3.9 holds with

$$b = \|F^{/}(x_0)\| + Kr \quad (2.12)$$

It should be noted that under the hypothesis of the foregoing proposition, the convergence of the Newton sequence $\{x_n\}$ implied the existence of a solution $x = x^*$ of $F(x) = 0$.

It follows that if the existence and convergence of Newton sequence can be established, Hhen it is certain that the equation (1.1) has a solution.

Nest, we give the statement of the theorem of convergence of Newton's method in Banach spaces as formulated by the kantrovich. Theorem below.

Consider the Newton sequence $(x_n\}$ sating from some point x_0. it is assumed that $[F^1(x_0)]^{-1}$ exists and permits the calculation of the next point,

$$x_1 = x_0 - [F^{/}(x_0)]^{-1} F(x_0)] \quad (2.13)$$

and there exists constants B_0 , n_0 such that

$$\| [F(x_0)]^{-1} \| \leq B_0 \quad (2.14)$$

$$\|x_1-x_0\| \leq n_0 \quad (2.15)$$

respectively

Theorem 4 (Kantrovich's theorem) Rall (1969),Argyros (2005):

$$IF \| F^{//}(x) \| \leq k \quad (2.16)$$

In some closed ball $\bar{B}(x_0, r)$ and $h_0 = B_0 \eta_0 k \leq ½$,

The Newton sequence $\{x_n\}$, converges to a solution x^* of equation (1.1) which exist in $B(x_0,r)$ provided that

$$r \geq r_0 = \frac{1-\sqrt{(1- 2h_0)}}{h_0} \eta_0 \quad (2.17)$$

Modified Newton's method:

Methods for computing the solutions of nonlinear operator equations are related in some way to Newton's method or method of successive approximations.

We consider the sequence,

$$X_{n+1} = x_n- L_n^{-1}y_{n.} \quad (3.1)$$

Where , L_n^{-1} is an approximations of $[F'(x_n)]^{-1}$ and y_n is close to $F(x_n)$ respectively could be used to describe a variant of Newton's method.

A similar process leads to the modified Newton's method,

$$X_{n+1} = x_n - [F'(x_0)]^{-1} F(x_n) \quad (3.2)$$

This procedure has as advantage of the reduction in labour of calculating $F^{/}(x_0)$ and its inverse $[F^{/}(x_0)]^{-1}$ since this is done once and for all, after which the fixed point of the modified Newton iteration procedure is sought by successive approximations.

Existence of Solutions of Fredhom Nonlinear Integral Equations by Newtons Method:

In this section, we consider the existence of solution for a nonlinear integral equation of the type,

$$x(s) = f(s) + \lambda \int_a^b K(s,t)\, x(t)^p\, dt,\ s \in (a, b],\ p \geq 2 \qquad (4.1)$$

where λ is a real number, the kernel K(s,t) and f(s) are numbers also following the analysis of Guitierrez. *et al.,* (2004), we express (4.1) in the form

$$F(x) = 0 \qquad (4.2)$$

where F: Ω C _X ➔ Y is a nonlinear operator defined by

$$F(x)(s) = x(s) - f(s) - \lambda \int_a^b K(s,t)\, x(t)^p\, dt,\ p \geq 2 \qquad (4.3)$$

and X = Y = C[a,b] is the space of continuous function on the interval [a, b] equipped with the max norm.

$\|x\| = \max\{|x(s)| : s \in [0,1]\}$.

We apply the modified form of Newton's method defined by

$$x_{n+1} = x_n - T_0 F(x_n),\ n \geq 0 \qquad (4.4)$$

where T_0 is the inverse of the linear operator defined from X to Y by

$$F'(x_0)y(s) = y(s) - \lambda p \int_a^b K(s,t) x(t)^{p-1} y(t)\, dt,\ x \in [a,b],\ y_\epsilon X \qquad (4.5)$$

Let $N = \max \int_a^b |K(s,t)|\,dt$ and x_0 be a function in such that $T_0 = [F'(x_0)]^{-1}$

exists and $\|T_0 F(x_0)\| < \eta$

$< \eta$ and $M = |\lambda| pN$.

Next, we assume the following conditions, Gutierrez *et al.,* (2004)

$$\left.\begin{array}{l} \text{(i) } \eta < R, R \text{ a real number} \\ \text{(ii) } a = M(\|x_0\| + R)^{p-1} < 1 \\ \text{(iii) If we denote } b = \dfrac{(p-1)\eta}{2(\|x_0\| + R)} \text{ and } h(t) = \dfrac{1}{1-t} \\ \text{Than abh (a)<1} \\ \text{(iv) } 2(\eta - t) + M(\|x_0\| + t)^{p-2}[(p-1)\eta t - 2(\eta - t)(\|x_0\| + t) = 0. \end{array}\right\} \qquad (4.6)$$

We will consider the following Lemmas:

Lemma 5 Ezquerro and Hermandez (2004):

From (i) to (iv) of (4.6) , it follows that

$$\sum_{i=0}^{n} (abh(a))^i\, \eta < \frac{\eta}{1 - abh(a)} = R$$

Theorem 6 (Banach Lemma on Invertible Operators), Argyros (2005):

If T is a bounded linear operation in X, T^{-1} exists if and only if there is a bounded linear operator P in X such that P^{-1} exists and $\| 1 - PT \| < 1$.

LEMMA 7 Guitierrez. et al (2004) evanglies:

If B (x_0, R) C _ Ω, then for all x C B (x_0, R), $[F'(x)]^{-1}$ exists and $\|[F'(x)]^{-1}\| \leq h(a)$.

Proof:

Applying theorem 3.13, and noting that

$$(1 - F'(x))\, y(s) = \lambda p \int_a^b K(s,t)\, x(t)^{p-1}\, y(t)\, dt,$$

Then,

$$\| 1 - F'(x) \| \leq |\lambda| pN \|x\|^{p-1}$$
$$\leq M (\|x_0\| + R)^{p-1} = a < 1$$

Therefore, $[F'(x)]^{-1}$ exist and $\| F'(x)^{-1} \| \leq \dfrac{1}{1-a} = h(a)$

Applying Theorem 6, and noting that $(1-F'(x))\, y(s) = \lambda p \int_a^b K(s,t) x(t)^{p-1} y(t)dt$.

Then,

$$\|1-F'(x)\| \leq |\lambda| pN \| x\|^{p-1}$$
$$\leq M (\|x_0\| + R)^{p-1} = a < 1.$$

Therefore, $[F'(x)]^{-1}$ exists and $\|F'(x)]^{-1}\| \leq \dfrac{1}{1-a} = h(a)$.

The above Lemma guaratees the existence of T_0 for some $x_0 \in X$, where T_0 is the inverse of the linear operator F (x_n) at n = 0

Lemma 8 Guitierrez et al., (2004):

If B $(x_0, R) \subseteq \Omega$ and assumptions (i) to (iv) of (4.6) hold,

Then for $x_n, x_{n-1} \in B(x_0,R)$

$$\|F(x_n)\| \leq \tfrac{1}{2} M(p - 1) (\|x_0\| + R)^{p-2} \|x_n - x_{n-1}\|^2$$

Proof:

Using Taylor's formula , we have

$$F(x_n)(s) = \int_0^1 [F'(x_{n-1} + y(x_n - x_{n-1})) - F'(x_n - 1)](x_n - x_{n-1})(s)\, dy$$

$$= - \lambda p \int_0^1 \int_a^b k(s,t) \quad p_n (y,t)^{p-1}\, x_{n-1}(t)^{p-1}] (x_n(t) - x_{n-1}(t)]dtdy$$

$$= - \lambda p \int_0^1 \int_a^b K(s,t) \left[\sum_{i=0}^{p-2} p_n (y,t)^{p-2-i}\, x_{n-1}(t) \right] [x_n(t) - x_{n-1}(t)]^2\, ydtdy$$

Where $p_n(y,t) = x_{n-1}(t) + y\ (x_n - x_{n-1})$ and we have used the inequality,

$$a^{p-1} - b^{p-1} = \left(\sum_{i=0}^{p-2} a^{p-2-1} b^{1}\right)(a - b)\ a,b \in R.$$

Since x_{n-1}, $x_n \in B\ (x_0,R)$ for each $y \in [0,1]$, $p_n\ (y,,) \in B(x_0, R)$,
Then,

$\| p_n\ (y)\| \leq \|x_0\| + R$. Consequently,

$$\| F(x_n)\| \leq \frac{|\lambda| p\ N}{2} \left(\sum_{i=0}^{p-2}(\| x_0 \| + R)^{p-2-i} \| x_{n-1}\|^{i}\right) \|x_n - x_{n-1}\|^2$$

$$\leq \frac{|\lambda| p(p-1)}{2} [\ \|x_0\| + R\ ]^{p-2}\ \| x_n - x_{n-1}\|^2$$

$$= \frac{(p-1)\ a}{2(\|x_0\| + R)} \|x_n - x_{n-1}\|^2 \tag{4.7}$$

$$= \frac{1}{2} M(P-1)\ (\|x_0\| + R)^{p-2} \| x_n - x_{n-1}\|^2 \tag{4.8}$$

Which completes the proof.

Next, we give the result on the existence of a solution of equation (4.2). This theorem is a modified form of Guitierrez et al (2004), and can be stated as follows.

Theorem 9:

Let (4.6) (iv) have a positive solution with $t = R$ being smaller solution and let $B\ (x_o, R) \subseteq \Omega$, then (4.2) has least one solution $x^* \in B\ (x_0,R)$.

Proof:

The proof of this Theorem follows from Guitierrez et al with some modifications.
We note that.

$X_{0+1} = x_n - T_n F(x_n) = x_n - [F(x_n)]^{-1} Fx_n$

$x_1 = x_0 - T_0 F(x_0)$, $\|T_0 F(x_0)\| < \eta$

so $\|x_1 - x_0\| = \|T_0 F(x_0\| < \eta < R$ by (4.6) (i).

and $x_1 \in B\ (x_0, R)$.

From (4.7)

$$\|F\ (x_1)\| \leq \frac{(p-1)a}{2(\|x_0\| + R)} \|x_1 - x_0\|^2 = ab\eta$$

And therefore
$\| x_2 - x_0\| \leq abh(a)\ \eta$

Then by Lemma 5
$\| x_2 - x_0\| < \| x_2 - x1\| + \| x_1 - x_0\|$

$\leq abh\ (a)\ \eta + \eta$
$\leq (1 + abh\ (a))\ \eta < R,$

It follows that $x_2 \epsilon B(x_0, R)$, By induction, we have that
$\| x_n - x_{n-1} \| \le (abh(a))^{n-1} \| x_1 - x_0 \|$ (4.9)

In addition by triangle inequality, we also have

$$\| x_n - x_0 \| \le \left(\sum_{i=0}^{n-1} (abh(a))^1 \right) \| x_1 - x_0 \| < \left(\sum (\| (abh(a)^1 \right) \eta < \frac{\eta}{1-abh}$$

Consequently, $x_n \epsilon B(x_0, R)$ for all $n \ge 0$. Next we prove that $\{x_n\}$ is a Cauchy sequence.
From (4.6) (i) to (iv) and (4.9).

$$\|x_{n+m} - x_n\| \le \| x_{n+m} - x_{n+m\ 1} \| + \| x_{n+m}\ 1 - x_{n+m\ 2} \| + \ldots + \| x_n - x_{n-1} \|$$

$$\le [(abh(a)^{m+n-1}\ {}_1(abh(a))^{m-2} + \ldots + a^{-1}\ bh\ (a)\ ^n] \|x_1 - x_0\|$$

$$=(abh(a))^n \quad \frac{1 - (abh(a))}{1 - abh(a)} \quad \eta$$

then, by letting $m \to \infty$, we have $\|x^* - x_0\| \le (abh(a))^n \dfrac{\eta}{1\ abh(a)}$

Since $\lim_{m \to \infty} x_m = x^*$ and $abh(a) < 1$.

Finally, for $n = 0$

$$\|x^* - x_0\| < \frac{\eta}{1\ abh(a)}$$

And $x^* \epsilon B(x_0, R)$. also from (4.9)
$\|F(x_n)\| \le \frac{1}{2} M (p\ 1)(\| x_0 \| + R)^{p-2} \| x_n\ x_{n-1} \|^2$

As $n \to \infty$, we obtain $F(x^*) = 0$ and x^* is a solution of $F(x) = 0$.

Application of Newton's Method to Fredholm Nonlinear Integral Equation.:

We illustrate the theoretical result of section 4 with the following example consider the nonlinear integral equation of fedholm types and second kind.

$$X(s)\ 0.075\sin(\pi s) + 1/5 \int_0^1 \cos(\pi s)\sin(\pi s)\ x(t)s\ dt,\ s\ \epsilon\ [0,1] \tag{5.1}$$

Let X = C[0,1] be a space of continuous function defined on the interval (0,1) with the max norm and let

F: X X be an operator defined by.

$$F(x)(s) = x(s) - 0.075 \sin(\pi s) + 1/5 \int_0^1 \cos(\pi s) \sin(\pi s) x(t)^3\ dt,\ s\ \epsilon\ [0,1] \tag{5.2}$$

By differentiating (5.2)

$$F^{/}(x)\ y(s) = y(s) - 3/5 \cos(\pi s) \int_0^1 \sin(\pi t) x(t)^2\ y(t)\ dt \tag{5.3}$$

Which by section 4 gives.

$$\lambda = 1/5,\ N = \max \int_0^1 |\sin(\pi t)|\,dt = 1,\ M = \lambda pN = 3/5$$

Choosing $x_0(s) = 0$ as a starting point,

$\|F(x_0)\| = 0.075$, and from (5.3)
$F^{/}(x_0)\, y(s) = y(s)$ giving,

$$F^{/}(x_0) = 1 \tag{5.4}$$

From section 4
To verify $[\ x_0)]^{-1}$ assuming that To $\leq \|T_0\| = 1$ then
$\|T_0F(x_0)\| \leq 0.075 = \eta$

Now, the equation (4.6) (iv) becomes.
$1.2t^3 - 2t + 0.15 = 0$
This equation has two positive roots , the smaller one is R = 0.075255….
By theorem 6, (5.1) has a solution in the ball B (x_0, R).
From (4.4), we defined the modified Newton's method by
$X_{n+1}(s) = x_n(s) - T_0 F(x_n)$, $n \geq 0$…5.5

With the function $x_0(s)$ as a starting point. First, we set

$$A_n = \int_0^1 \sin(\pi t)\, x_n(t)^3\, dt$$

Then

$X_{n-1} = 0.075\sin(\pi s) + 1/5\, A_n \cos(\pi s)$,
And we obtain the following approximations :
$X_1(s) = 0.075\sin\pi s$.
$X_2(s) = 0.075\sin\pi s + 3.1640625\text{x}10^{-5}\cos\pi s$
$X_3(s) = 0.075\sin\pi s + 3.1640306\text{x}10^{-5}\cos\pi s$
$X_4(s) = 0.075\sin\pi s + 3.16406306\text{x}10^{-5}\cos\pi s$
this implies that the modified Newton's method converges to the solution
$x^*(s) = 0.075\sin\pi s + 3.16406306\text{x}10^{-5}\cos\pi s$.

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