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ORIGINAL ARTICLE

Beautiful Algebra

S. Kalimuthu

211, Kanjampatti P.O, PollachiVia, Tami Nadu 642003, India.

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ABSTRACT

It has been established mathematically that it is impossible to deduce Euclid V from Euclid I to IV But,.in this work,by applying the fundamental operations of number theory and algebra, the author has found a consistent solution which may give rise to a new field of science.

Key words: Euclid,elements,postulates,triangles,angles,number theory,algebra

Journey through Euclid

Euclid's 5th Postulate states that lines will always intersect at some point unless they are parallel. However, this is an *axiom*, not a *theorem*. In other words, Euclid just assumed this to be a geometric truth, without proof. Many subsequent mathematicians believed this Postulate was independent of the other 4 Postulates; one could prove it as a Theorem using only the other Postulates. However, nobody was ever able to complete such a proof, and in 1868, the mathematician Beltrami formally proved that the 'Axiom of Parallels' was completely independent of the other Postulates.

What does this mean? Apart from Euclid's Postulate, there is no guarantee that parallel lines cannot meet. Thus the several varieties of 'non-Euclidean' Geometry (where parallel lines can meet) can be entirely consistent.

Why did mathematicians feel the need to deduce the parallel postulate from the other axioms of geometry? After all, if you are going to start from some axioms, it doesn't much matter how many there are. Nobody seemed to mind that the other axioms were independent of each other.

Suspicion of the parallel postulate goes back to Euclid, who was the first person to notice (in writing at any rate) that it was needed for some arguments. Whenever he could, he avoided using it, even if that meant producing longer proofs. Did people have some inkling of non-Euclidean geometry, some premonition that the parallel postulate might be false?

No, they certainly did not. However, they felt uneasy about the parallel postulate because it was more complicated to state than the other axioms, and not quite as obviously true. If you have a line L and a point x not on it and claim that there is a line M through x that does not meet L, then you are making a statement about the whole, infinite line M and are therefore on dodgier ground than you are with the other axioms. It seems strange to have to deduce that the angles of a triangle add to 180 by appealing to what goes on unboundedly far away.

Incidentally, there were some serious attempts at proofs of the parallel postulate, but they all turned out to depend on hidden assumptions that were themselves equivalent to the parallel postulate (as is obvious if one bears hyperbolic geometry in mind). For example, one proof used the fact that for every triangle there is a similar triangle of any given size - which is false in the hyperbolic plane.

The development of non-Euclidean geometry caused a profound revolution, not just in mathematics, but in science and philosophy as well.

Corresponding Author: S. Kalimuthu, 211, Kanjampatti P.O,PollachiVia,TamiNadu642003,India
Email: postulate.kalimuthu0@gmail.com
Mobile: 91 97885 49996

The philosophical importance of non-Euclidean geometry was that it greatly clarified the relationship between mathematics, science and observation. Before hyperbolic geometry was discovered, it was thought to be completely obvious that Euclidean geometry correctly described physical space, and attempts were even made, by Kant and others, to show that this was necessarily true. Gauss was one of the first to understand that the truth or otherwise of Euclidean geometry was a matter to be determined by experiment, and he even went so far as to measure the angles of the triangle formed by three mountain peaks to see whether they added to 180. (Because of experimental error, the result was inconclusive.) Our present-day understanding of models of axioms, relative consistency and so on can all be traced back to this development, as can the separation of mathematics from science.

The scientific importance is that it paved the way for Riemannian geometry, which in turn paved the way for Einstein's General Theory of Relativity. After Gauss, it was still reasonable to think that, although Euclidean geometry was not necessarily true (in the logical sense) it was still empirically true: after all, draw a triangle, cut it up and put the angles together and they will form a straight line. After Einstein, even this belief had to be abandoned, and it is now known that Euclidean geometry is only an approximation to the geometry of actual, physical space. This approximation is pretty good for everyday purposes, but would give bad answers if you happened to be near a black hole, for example.

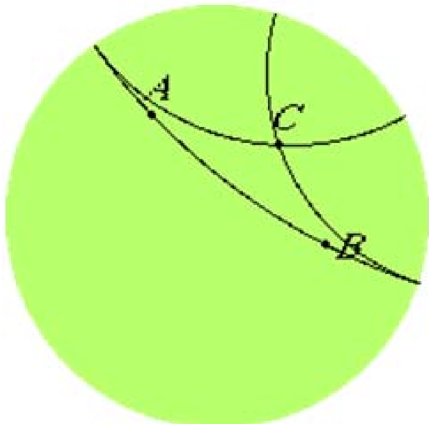
Even before Beltrami proved the independence of the Parallel Postulate, mathematicians were still able to work on Projective Geometry. In the early 17^{th} Century, Kepler suggested the notion of 'points at infinity' where parallel lines would intersect; meanwhile Desargues and Pascal began to study Geometry using only intersections. Once Kepler's idea was taken seriously, Geometers saw that the Geometry of intersections (incidence relations) could be made into a wholly consistent theory. As suggested above, if all lines are guaranteed to meet at one point, the study of intersections does not have to make any exceptions (a flaw of Euclidean Geometry). Finally, in 1871, Klein proved that the entire theory of Projective Geometry is independent of the Parallel Postulate.

Hyperbolic geometry

Two important geometries alternative to Euclidean geometry are elliptic geometry and hyperbolic geometry..

These three geometries can be distinguished by the number of lines parallel to a given line passing through a given point. For elliptic geometry, there is no such parallel line; for Euclidean geometry (which may be called parabolic geometry), there is exactly one; and for hyperbolic geometry, there are infinitely many.

It is not possible to illustrate hyperbolic geometry with correct distances on a flat surface since a flat surface is Euclidean. Poincaré, however, described a useful model of hyperbolic geometry where the "points" in a hyperbolic plane are taken to be points inside a fixed circle (but not the points on the circumference). The "lines" in the hyperbolic plane are the parts of circles orthogonal, that is, at right angles to the fixed circle. And in this model, "angles" in the hyperbolic plane are angles between these arcs, or, more precisely, angles between the tangents to the arcs at the point of intersection. Since "angles" are just angles, this model is called a *conformal* model. Distances in the hyperbolic plane, however, are not measured by distances along the arcs. There is a more complicated relation between distances so that near the edge of the fixed circle a very short arc models a very long "line."



Once this model is accepted, it is easy to see why there are infinitely many "lines" parallel to a given "line" through a given "point." That is just that there are infinitely many circles orthogonal to the fixed circle which don't intersect the given circle orthogonal to the fixed circle but do pass through the given point.

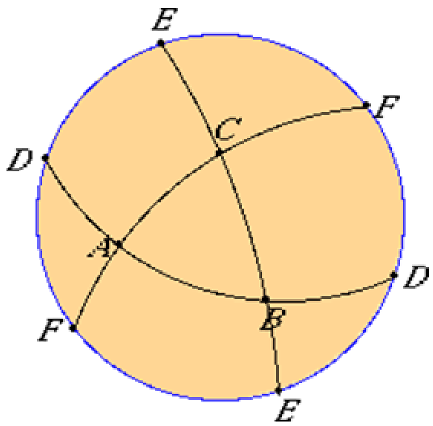
In the diagram, *AB* is a "line" in the hyperbolic plane, that is, a circle orthogonal to the circumference of the shaded disk which represents the hyperbolic plane. A "point" *C* lies in that plane. Two "lines" are shown passing through *C*, one gets close to the line *AB* in the direction of *A*, the other gets close in the direction of *B*. But these two "lines" don't intersect *AB* since the arcs representing them only intersect on the circumference of the disk, and points on the circumference don't represent "points" in the hyperbolic plane.

These two parallel "lines" are called the *asymptotic* parallels of *AB* since they approach *AB* at one end or the other. There are infinitely many parallels between them. (In much of the literature on hyperbolic geometry, the word "parallels" is used for what are called "asymptotic parallels" here, while "nonintersecting lines" is used for what are called "parallels" here.)

Elliptic geometry

Plane elliptic geometry is closely related to spherical geometry, but it differs in that antipodal points on the sphere are identified. Thus, a "point" in an elliptic plane is a pair of antipodal points on the sphere. A "straight line" in an elliptic plane is an arc of great circle on the sphere. When a "straight line" is extended, its ends eventually meet so that, topologically, it becomes a circle. This is very different from Euclidean geometry since here the ends of a line never meet when extended.

The illustration on the right shows the stereographic projection of one hemisphere. Since only one hemisphere is displayed, each "point" is represented by one point except those "points" such as *D*, *E*, and *F* on the blue bounding great circle which appear twice.



A "triangle" in elliptic geometry, such as *ABC*, is a spherical triangle (or, more precisely, a pair of antipodal spherical triangles). The internal angle sum of a spherical triangle is always greater than 180°, but less than 540°, whereas in Euclidean geometry, the internal angle sum of a triangle is 180° as shown in Proposition I.32. Elliptic geometry satisfies some of the postulates of Euclidean geometry, but not all of them under all interpretations. Usually, Post.1, to draw a straight line from any point to any point, is interpreted to include the uniqueness of that line. But in elliptic geometry a completed "straight line" is topologically a circle so that any pair of points on it divide it into two arcs. Therefore, in elliptic geometry exactly two "straight lines" join any two given "points."

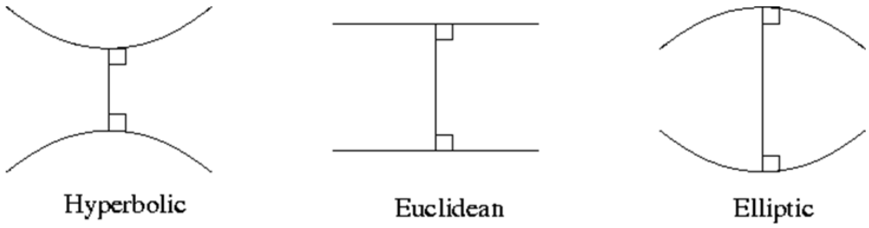
Also, Post.2, to produce a finite straight line continuously in a straight line, is sometimes interpreted to include the condition that its ends don't meet when extended. Under that interpretation, elliptic geometry fails Postulate 2.

Elliptic geometry fails Post.5, the parallel postulate, as well, since any two "straight lines" in an elliptic plane meet. That is, any two great circles on the sphere meet at a pair of antipodal points.

Finally, a completed "straight line" in the elliptic plane does not divide the plane into two parts as infinite straight lines do in the Euclidean plane. A completed "straight line" in the elliptic plane is a great circle on the sphere. Any two "points" not on that "straight line" include two points in the same hemisphere, and they can be joined by an arc that doesn't meet the great circle. Therefore two "points" lie on the same side of the completed "straight line."

The proof of this particular proposition fails for elliptic geometry, and the statement of the proposition is false for elliptic geometry. In particular, the statement "the angle *ECD* is greater than the angle *ECF*" is not true of all triangles in elliptic geometry. The line *CF* need not be contained in the angle *ACD*. All the previous propositions do hold in elliptic geometry and some of the later propositions, too, but some need different proofs.

Another way to describe the differences between these geometries is as follows: consider two lines in a plane that are both perpendicular to a third line. In Euclidean and hyperbolic geometry, the two lines are then parallel. In Euclidean geometry, however, the lines remain at a constant distance, while in hyperbolic geometry they "curve away" from each other, increasing their distance as one moves farther from the point of intersection with the common perpendicular. In elliptic geometry, the lines "curve toward" each other, and eventually intersect; therefore no parallel lines exist in elliptic geometry.



Behavior of lines with a common perpendicular in each of the three types of geometry

A btrief history of algebra

Algebra may divided into "classical algebra" (equation solving or "find the unknown number" problems) and "abstract algebra", also called "modern algebra" (the study of groups, rings, and fields). Classical algebra has been developed over a period of 4000 years. Abstract algebra has only appeared in the last 200 years. The development of algebra is outlined in these notes under the following headings: Egyptian algebra, Babylonian algebra, Greek geometric algebra, Diophantine algebra, Hindu algebra, Arabic algebra, European algebra since 1500, and modern algebra. Since algebra grows out of arithmetic, recognition of new numbers - irrationals, zero, negative numbers, and complex numbers - is an important part of its history.

The development of algebraic notation progressed through three stages: the rhetorical (or verbal) stage, the syncopated stage (in which abbreviated words were used), and the symbolic stage with which we are all familiar.

The materials presented here are adapted from many sources including Burton, Kline's Mathematical Development From Ancient to Modern Times, Boyer's A History of Mathematics , and the essay on "The History of Algebra" by Baumgart in Historical Topics for the Mathematics Classroom - the 31st yearbook of the N.C.T.M.

Egyptian Algebra

Much of our knowledge of ancient Egyptian mathematics, including algebra, is based on the Rhind papyrus. This was written about 1650 B.C. and is thought to represent the state of Egyptian mathematics of about 1850 B.C. They could solve problems equivalent to a linear equation in one unknown. Their method was what is now called the "method of false position." Their algebra was rhetorical, that is, it used no symbols. Problems were stated and solved verbally.

The Cairo Papyrus of about 300 B.C. indicates that by this time the Egyptians could solve some problems equivalent to a system of two second degree equations in two unknowns. Egyptian algebra was undoubtedly retarded by their cumbersome method of handling fractions.

Babylonian Algebra

The mathematics of the Old Babylonian Period (1800 - 1600 B.C.) was more advanced that that of Egypt. Their "excellent sexagesimal [numeration system]. . . led to a highly developed algebra" [Kline]. They had a general procedure equivalent to solving quadratic equations, although they recognized only one root and that had to be positive. In effect, they had the quadratic formula. They also dealt with the equivalent of systems of two equations in two unknowns. They considered some problems involving more than two unknowns and a few equivalent to solving equations of higher degree.

There was some use of symbols, but not much. Like the Egyptians, their algebra was essentially rhetorical. The procedures used to solve problems were taught through examples and no reasons or explanations were given. Also like the Egyptians they recognized only positive rational numbers, although they did find approximate solutions to problems which had no exact rational solution.

Greek Geometrical Algebra

The Greeks of the classical period, who did not recognize the existence of irrational numbers, avoided the problem thus created by representing quantities as geometrical magnitudes. Various algebraic identities and constructions equivalent to the solution of quadratic equations were expressed and proven in geometric form. In content there was little beyond what the Babylonians had done, and because of its form geometrical algebra was of little practical value. This approach retarded progress in algebra for several centuries. The significant achievement was in applying deductive reasoning and describing general procedures.

Diophantine Algebra

The later Greek mathematician, Diophantus (fl. 250 A.D.), represents the end result of a movement among Greeks (Archimedes, Apollonius, Ptolemy, Heron, Nichomachus) away from geometrical algebra to a treatment which did not depend upon geometry either for motivation or to bolster its logic. He introduced the syncopated style of writing equations, although, as we will mention below, the rhetorical style remained in common use for many more centuries to come.

Diophantus' claim to fame rests on his Arithmetica, in which he gives a treatment of indeterminate equations - usually two or more equations in several variables that have an infinite number of rational solutions. Such equations are known today as "Diophantine equations". He had no general methods. Each of the 189 problems in the Arithmetica is solved by a different method. He accepted only positive rational roots and ignored all others. When a quadratic equation had two positive rational roots he gave only one as the solution. There was no deductive structure to his work.

Hindu Algebra

The successors of the Greeks in the history of mathematics were the Hindus of India. The Hindu civilization dates back to at least 2000 B.C. Their record in mathematics dates from about 800 B.C., but became significant only after influenced by Greek achievements. Most Hindu mathematics was motivated by astronomy and astrology. A base ten, positional notation system was standard by 600 A.D. They treated zero as a number and discussed operations involving this number.

The Hindus introduced negative numbers to represent debts. The first known use is by Brahmagupta about 628. Bhaskara (b. 1114) recognized that a positive number has two square roots. The Hindus also developed correct procedures for operating with irrational numbers.

They made progress in algebra as well as arithmetic. They developed some symbolism which, though not extensive, was enough to classify Hindu algebra as almost symbolic and certainly more so than the syncopated algebra of Diophantus. Only the steps in the solutions of problems were stated; no reasons or proofs accompanied them.

The Hindus recognized that quadratic equations have two roots, and included negative as well as irrational roots. They could not, however, solve all quadratics since they did not recognize square roots of negative numbers as numbers. In indeterminate equations the Hindus advanced beyond Diophantus. Aryabhata (b. 476) obtained whole number solutions to $ax \pm by = c$ by a method equivalent to the modern method. They also considered indeterminate quadratic equations.

Arabic Algebra

In the 7th and 8th centuries the Arabs, united by Mohammed, conquered the land from India, across northern Africa, to Spain. In the following centuries (through the 14th) they pursued the arts and sciences and were responsible for most of the scientific advances made in the west. Although the language was Arabic many of the scholars were Greeks, Christians, Persians, or Jews. Their most valuable contribution was the preservation of Greek learning through the middle ages, and it is through their translations that much of what we know today about the Greeks became available. In addition they made original contributions of their own.

They took over and improved the Hindu number symbols and the idea of positional notation. These numerals (the Hindu-Arabic system of numeration) and the algorithms for operating with them were transmitted to Europe around 1200 and are in use throughout the world today.

Like the Hindus, the Arabs worked freely with irrationals. However they took a backward step in rejecting negative numbers in spite of having learned of them from the Hindus.

In algebra the Arabs contributed first of all the name. The word "algebra" come from the title of a text book in the subject, Hisab al-jabr w'al muqabala, written about 830 by the astronomer/mathematician Mohammed ibn-Musa al-Khowarizmi. This title is sometimes translated as "Restoring and Simplification" or as "Transposition and Cancellation." Our word "algorithm" in a corruption of al-Khowarizmi's name.

The algebra of the Arabs was entirely rhetorical.

They could solve quadratic equations, recognizing two solutions, possibly irrational, but usually rejected negative solutions. The poet/mathematician Omar Khayyam (1050 - 1130) made significant contributions to the solution of cubic equations by geometric methods involving the intersection of conics.

Like Diophantus and the Hindus, the Arabs also worked with indeterminate equations.

European Algebra after 1500

At the beginning of this period, zero had been accepted as a number and irrationals were used freely although people still worried about whether they were really numbers. Negative numbers were known but were not fully accepted. Complex numbers were as yet unimagined. Full acceptance of all components of our familiar number system did not come until the 19th century. Algebra in 1500 was still largely rhetorical. Renaissance mathematics was to be characterized by the rise of algebra.

In the 16th century there were great advances in technique, notably the solution of the cubic and quartic equations - achievements called by Boyer "perhaps the greatest contribution to algebra since the Babylonians learned to solve quadratic equations almost four millennia earlier." Publication of these results in 1545 in the Ars Magna by Cardano (who did not discover them) is often taken to mark the beginning of the modern period in mathematics. Cardano was the best algebraist of his age, but his algebra was still rhetorical. Subsequent efforts to solve polynomial equations of degrees higher than four by methods similar to those used for the quadratic, cubic, and quartic are comparable to the efforts of the ancient Greeks to solve the three classical construction problems: they led to much good mathematics but only to a negative outcome.

There were also at this time many important improvements in symbolism which made possible a science of algebra as opposed to the collection of isolated techniques ("bag of tricks") that had been the content of algebra up to this point.

The landmark advance in symbolism was made by Viète (French, 1540-1603) who used letters to represent known constants (parameters). This advance freed algebra from the consideration of particular equations and thus allowed a great increase in generality and opened the possibility for studying the relationship between the coefficients of an equation an the roots of the equation ("theory of equations"). Viète's algebra was still syncopated rather than completely symbolic. Symbolic algebra reached full maturity with the publication of Descartes' La Géométrie in 1637. This work also gave the world the wonderfully fruitful marriage of algebra and geometry that we know today as analytic geometry (developed independently by Fermat and Descartes). "By the end of the 17th century the deliberate use of symbolism - as opposed to incidental and accidental use - and the awareness of the power and generality it confers [had] entered mathematics." [Kline] But logical foundations for algebra comparable to those provided in geometry by Euclid were nonexist.

As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection."-- Joseph-Louis Lagrange

Keeping this in mind,let us transform the properties of spherical triangles and quadrilaterals into linear algebraic equations and derive an exciting result

Construction

Construct a triangle ABC.On BC, choose two points G and H.Join A and H;Join A and G.On AB,take two points D and E.On AC make a point F.Join E and F contacting AG at J and AH at K.Since points E and F lie on the opposite sides of AG and AH EF can meet AG and AH.Please note that Euclid uses this principle .[1,prop.10]Similarly join C and D meeting JG at T and KH at L.Small letters denote the sum of the interior angles in triangles and quadrilaterals.Also, let that the sum of the interior angles of triangles and quadrilaterals, AEF = a, AEK = b, AJF= c, DEFC = d, JTCF = e, DEKL = f, BCD = g, BHLD = h, CGT = j, ABG = k, AHG = l, ACH = t, ADT = u, EBGJ = w, ATL = a', JGHK = b', ALC = c', KFCH = d'

Result

The angles,EJK,JKF,DTL,TLC,BGH,GHC,AED,EDB,AJT,JTG,AKl,KLH and AFC are all straifgt angles

and so their measures are all equal to 180 degrees.Let v be the value of this . (1)
Assuming (1), $x + y + z = 2v + a$ (2)
$v + b = x + y$ (3)
$y + z = v + c$ (4)
$4v + d = m + n + p$ (5)
$m + n = 2v + e$ (6)
$p + n = 2v + f$ (7)
$4v + g = q + r + s$ (8)
$q + r = 2v + h$ (9)
$2v + j = s + r$ (10)
$x + p + q = 4v + k$ (11)
$4v + l = y + n + r$ (12)
$z + m + s = 3v + t$ (13)
$2v + u = x + p$ (14)
$p + q = 2v + w$ (15)
$2v + a' = y + n$ (16)
$n + r = 2v + b'$ (17)
$2v + c' = z + m$ (18)
$m + s = v + d'$ (19)

Adding (2) to (19), $b + 2z + d + g + j + q + l + u + p + q + a' + m = a + c + e + f + r + h + K + t + w + y + b' + d'$
Adding s to both sides, $s = s$
Putting (3) in (2), $v + a = b + z$
$2v + c' = m + z$ (18)
$2v + w = p + q$ (15)
$s + r = 2v + j$ (10)
Applying (9) in (8), $h + s = 2v + g$
$2v + f = p + n$ (7)
$4v + k = x + p + q$ (11)
$3v + t = z + m + s$ (13)
$2v + b' = n + r$ (17)
$v + d' = m + s$ (19)
$y + n + r = 4v + l$ (12)
$x + p = 2v + u$ (14)
(14) + (18) gives, $y + z + m + n = 4v + a' + c'$
Adding the above fifteen eqns, $3v + y + c' = p + m + c$
$m + n + p = 4v + d$ (5)
$z + m = 2v + c'$ (18)
$v + c = y + z$ (4)
Adding the above four relations, $m + n = 2v + d$ (20)
Applying (20) in (5) we get that, $p = 2v$ (5a)
i.e the sum of the interior angles of quadrilateral
DEKL is equal to 360 degrees[from (1)] (21)

Dicsussion

Since we have derived (21) without assuming the parallel postulate. (21) establishes the fifth Euclidean postulate.[2 - 7] Our construction,i.e figure 1 can be extended to both hyperbolic and elliptic spaces also.Through out this work, we have applied only the fundamental operations of number theory and algebra.So, (21) is consistent.If it is inconsistent, immediately it implies that one plus two is NOT equal to three.This is absurd.Similarly to brand that (21) is incorrect is also absurd.Only God is the Number One expert.The almighty reveals some message through (21).We have to probe into (21) which will definitely give birth to a new field of science.

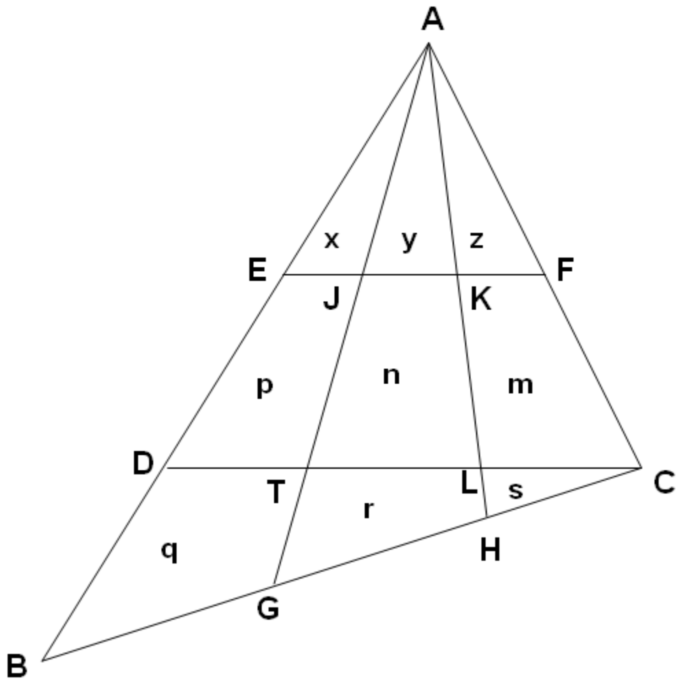


Fig. 1: Euclidean

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