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ORIGINAL ARTICLE

On the Existence of Solution of Nonlinear Equations by Method of Successive Approximations

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ABSTRACT

This paper provides convergence analysis for a type of fixed point iterations in Banach spaces. By modifying a contractive condition in (Ezquerro, J. and A. Hernandez, 2004), we obtain an error estimate that gives precise information of the location of solution for a nonlinear operator equation satisfying such condition. Finally an illustration is given with an application to a nonlinear integral equation of Fredholm type and second kind.

Key words: Banach spaces, Frechet differentiability, Pichard's iteration, radius of convergence, Fredholm integral equations.

In this paper, we want to established the existence of a solution Φ * of a nonlinear equation.

$$F(\Phi) = 0 \tag{1}$$

where F is a Frechet-differential operator defined on an open convex domain D of a Banach space X with values in a Banach space Y.
We consider the method of successive approximations

$$\Phi_{m+1} = \Phi_m - F(\Phi_m), m = 0,1,2 \dots \tag{2}$$

Which is also known as Picard iteration (Ezquerro, J. and A. Hernandez, 2004).
The Picard iteration operator for (2) can be defined as

$$G(\Phi) = \Phi - F(\Phi) \tag{3}$$

and

$$G'(\Phi) = I - F'(\Phi) \tag{4}$$

Let us assume the following conditions for $\Phi_0 \epsilon D$:

$$G'(\Phi_0) = I - F(\Phi_0) = 0, F(\Phi_0) \neq 0 \tag{5}$$

where I is an identity operator.

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$$I = F'(\Phi_0) \tag{6}$$

by (5)

$$\|F'(\Phi) - F(\Phi_0)\| \leq a\|\Phi - \Phi_0\|, a > 0 \tag{7}$$

and

$$\|F'(\Phi) - I\| \leq a\|\Phi - \Phi_0\| \tag{8}$$

by (6) and (7).

$$\|H(\Phi)\| = \|H(\Phi) - H(\Phi_0)\| \leq H'(\Phi)\|\Phi - \Phi_0\| \leq ar_0^2$$

Since

$$G(\Phi) - \Phi_0 = \left(H(\Phi) - F(\Phi_0)\right),$$

$$\|G(\Phi) - \Phi_0\| \leq \|H(\Phi) - H(\Phi_0)\| + \|F(\Phi_0)\| \leq ar_0^2 + \eta = r_0$$

Hence, $G(\bar{\mathrm{U}}) \subseteq D.$

Now for $\Phi \epsilon \bar{\mathrm{U}},$

$$G'(\Phi) = 1 - F'^{(\Phi)} = H(\Phi)$$

So that

$$\|G'(\Phi)\| \leq \omega(r_0) = ar_0 \leq ar_1 \leq 1.$$

Hence, G is a contraction on $\bar{\mathrm{U}}(\Phi_0, r_0)$.

Next, by Banach fixed point theorem (Krezig, E., 1979)
For m, n >=1

$$\|\Phi_{m+n} - \Phi_m\| = \frac{1-\omega(r_0)^n}{1-\omega(r_0)}\omega(r_0)^m\|\Phi_1 - \Phi_0\|$$

And since $\omega(r_0) < 1$, then $1 - \omega(r_0)^n < 1$

Thus,

$$\|\Phi_{m+n} - \Phi_m\| \leq \frac{\omega(r_0)^m}{1-\omega(r_0)}\|\Phi_1 - \Phi_0\|.$$

By letting $n \to \infty$.

$$\|\Phi_m - \Phi^*\| \leq \frac{\omega(r_0)^m}{1-\omega(r_0)}\|\Phi_1 - \Phi_0\|.$$

Finally for m=0,

$$||\Phi^* - \Phi_0|| \leq \frac{\eta}{1-\omega(r_0)} = r_0$$

This show that $\Phi^* \epsilon \mathrm{U}$ and $F(\Phi^*) = 0.$

2.0 Convergence of Method of Successive Approximations (2)

In this section, we give the convergence analysis of method of successive approximations (2).

2.1 Local Convergence

Lemma 2.1

Let $\Phi^* \epsilon D$ be a zero of F and suppose there exist r>0 such that

i) F is differentiable on an open ball $\mathrm{U}(\Phi^*, r) = \{\Phi^* \epsilon X: ||\Phi - \Phi^*|| < r\}$

ii) $\omega = ||1 - F'(\Phi)|| < 1$.

Then for all $\Phi_0 \epsilon \mathrm{U}(\Phi^*, r),$ the sequence defined by (Ahues, M., 2004) converges to Φ^*

Proof:

Let $\Phi_0 \epsilon \mathrm{U}(\Phi^*, r)$ and suppose $\Phi_m \epsilon \mathrm{U}(\Phi^*, r)$ where,

$$\Phi_m(t) = \Phi^* + t(\Phi_m - \Phi^*), t\epsilon[0,1]$$

Now,

$$F(\Phi_m) = F(\Phi_m) - F(\Phi^*)$$

And by (2) and Taylor's formula

$$||\Phi_{m+1} - \Phi^*|| \leq \omega ||\Phi_m - \Phi^*|| < r,$$

Thus, $\Phi_{m+1} \epsilon \mathrm{U}(\Phi^*, r).$ Since $\omega < 1$, the sequence defined by (2) converges to Φ^*

2.2 semilocal convergence

The result below gives condition for the existence of a unique solution of (1) for a given $\Phi_0 \epsilon D$ and also the convergence of the sequence defined by method of successive approximations (2).

Lemma 2.2

Suppose the following holds for $D, F, \Phi_0 \epsilon D, a > 0 \; and \; \eta$;

I) $\|\Phi_1 - \Phi_0\| \leq \eta$

ii) $\bar{\mathrm{U}} = \{\Phi \epsilon X : \|\Phi - \Phi_0\| \leq r_0\} \subseteq D \; and$

$\mathrm{U} = \{\Phi \epsilon X : \|\Phi - \Phi_0\| \leq r_0\} \subseteq D$

Where r_0 is the smallest root of the equation

$ar^2 - r + \eta = 0$

iii) $F' : \mathrm{U} \to L(X, X)$ exists and satisfy (7), then F has a unique square root Φ^* in U and for all $m \geq 0$

$\|\Phi_m - \Phi_0\| \leq \frac{\omega(r_0)^m}{1 - \omega(r_0)} \eta$

Where $0 \leq \omega(r_0) = ar_0 < 1.$

Proof:

Let $f : (0, \infty) \to \Re$ be defined by

$f(r) = (\omega(r) - 1)r + \eta.$

Observe that $f(0) = \eta \; and \; f' = 0$ for some $r_1 = \frac{1}{2a} \geq 0,$ at which

$f(r_1) = \eta - \frac{r_1}{2} \leq 0$ and $f(0)F(r_1) \leq 0$

Thus by intermediate value theorem, r_0 exists in the interval $0 < r_0 < r_1$

Let us define for $\Phi \epsilon \bar{\mathrm{U}}$,

$G(\Phi) = \Phi - F(\Phi)$

And define another operator.

$$H(\Phi) = F(\Phi_0) + (\Phi - \Phi_0) - F(\Phi), \quad (9)$$

see Ahues (2004), Argyros(2006),

Noting that $F(\Phi^*) = 0$ implies $\Phi^* = G(\Phi^*).$

By (9), $H(\Phi_0) = 0$ and for all $\Phi \epsilon \bar{\mathrm{U}}$

$H'(\Phi) = 1 - F'(\Phi)$

$$\|H'(\Phi)\| = \|1 - F'(\Phi)\| \leq ar_0$$

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