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ORIGINAL ARTICLE

Fixed Point and the Contraction Mapping Principle with Application to Nonlinear Integral Equation of Radiative Transfer

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ABSTRACT

Let F be an operator mapping a set X into itself. A point x ϵ X is called a fixed point of F if x = F(x). Hence finding a fixed point on an operator F is equivalent to obtaining a solution of f(x) = 0. By this research work, we consider the contraction mapping principle and its application in the solution of the non linear integral equation of radiative transfer.

Key words: The fixed point, contraction mapping, iterates, integral equations.

Introduction

1.0 definition

Let F be an operator mapping a set X into itself. A point x Є X is called a fixed point of F if

$$x = F(x) \qquad 1.1$$

Finding a fixed point of an operator F is equivalent to obtaining a solution of (1.1)

If the value x_0 of X is known such that $F(x_0)$ does not differ greatly from x_0, wee can naturally regard

$$x_1 = F(x_0) \qquad 1.2$$

As a probable improvement over x_0 and the equation (1.1) leads to the generation of the sequence $\{x_m\}$ of successive approximations to a fixed point x of F by the relationship

$$x_{m+1} = F(x_m)\ ,\ (m \geq 0) \qquad 1.3$$

The constructive procedure of generating the sequence $\{x_m\}$ is known as method of iteration.

Theory of the Contraction Mapping Principle

Theorem 2.1(the Contraction Mapping Principle) Rall (1969)

If F is an operator in a Banach space X which is a contraction mapping of $\overline{B}(x_0, r)$ for

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$$r \geq \frac{1}{1\text{-}a}\|x_0 - F(x_0)\| = r_0 \qquad 2.1$$

where **α** is a contraction factor for F on B (x_0, r) then,

F has a fixed point x^* in $\overline{B}$ (x_0, r)

x^* is the unique fixed point of F in $\overline{B}$ (x_0, r)

The sequence $\{x_m\}$ of successive approximations defined by (1.3)
Converges to x^* with

$$\|x_m - x^*\| \leq \alpha^m r_0 \qquad 2.2$$

Proof

First we show that the sequence $x_n \in \overline{B}$ (x_0, r)

As $x_1 = F(x_0)$ 2.3

Then from (2.1)

$$\|x_1 - x_0\| = (1 - \alpha) r_0 < r_0 \qquad 2.4$$

So $x_1 \in \overline{B}$ (x_0, r_0)

Assume that $x_0, x_1, \ldots x_{n-1}, x_n \in \overline{B}$ (x_0, r_0) and that

$$\|x_n - x_0\| \leq (1 - \alpha^n) r_o < r_0 \qquad 2.5$$

For some positive integer n
Then since

$$\|x_{n+1} - x_n\| = \|F(x_n) - F(x_{n-1})\| \leq \alpha \|x_0 - x_{n-1}\| \qquad 2.6$$

Successive application of (2.6) gives

$$\|x_{n+1} - x_n\| \leq \alpha^n \|x_1 - x_0\| = \alpha^n (1 - \alpha) r_0 \qquad 2.7$$

and

$$\|x_{n+1} - x_0\| \leq \|x_n - x_0\| + \|x_{n+1} - x_n\|$$

$$\leq (1 - \alpha^n) r_0 + \alpha^n (1 - \alpha) r_0 = (1 - \alpha^{n+1}) r_0 < r_0 \qquad 2.8$$

Hence $\{x_m\}$ is contained in $\overline{B}$ (x_0, r)

(b) Next, we show that $\{x_m\}$ is a Cauchy sequence and, thus has a limit point $x^* \in \overline{B}$ (x_0,r_0).

Observe that,

$$\|x_{m+p} - x_m\| \leq \|x_{m+p} - x_{m+p-1}\| + \|x_{m+p} - x_m\| \qquad \leq 1 - \alpha^p \qquad 2.9$$

$$1 - \alpha \qquad \|x_{m+1} - x_m\| \qquad 2.10$$

And by (2.7) we obtain,

$\|x_{m+p} - x_m | \leq (1 - \alpha^p)\alpha^m r_0$ 2.11

Thus the sequence $\{x_m\}$ is Cauchy . Hence, its limit x^* exist in $\overline{B}$ (x_0, r),

$\|x^* - x^0\|=\|F(x^*) - F(x^0)\| \leq \alpha\|x^*-x^0\|<\|x^* - x^0\|$ 2.12

Which is impossible.

Also from (2.11)by letting $p \longrightarrow \infty$ the error bound (2.2) is obtained
The proof is complete.
An immediate consequence of theorem (2.1) follows:
Suppose that bounded linear operators T, T^{-1}, L and a point $y \in X$ are given.
A solution of the linear equation.

$L(x) = y$ 2.12

Will be a fixed point of the operator F defined by

$F(x) = (I - TL) x + Ty$ 2.13

If $\alpha = \|I - TL\| < 1$ 2.14

Then by theorem 2.1, x^* exist such that

$x^* = (I - TL)x^* = Ty$ 2.15

that is, $Lx^* = y$ exists in the ball $B(x_0, r)$, where

$r_0 = (I - \|I - TL\|)^{-1}\| I - TL) x_0 + Ty\|$ 2.16

for any $x_0 \in X$ and x^* is unique in X

the error bound

$$\|x_m - x^*\| \leq \frac{\|I - TL\|^m}{I - \| I\text{-}TL\|}\| (I - TL)x_0 + Ty\| \qquad 2.17$$

Is obtained for

$x_m = (I - TL) x_{m-1} + Ty, \quad M = 1,2,3$ 2.18

We give an illustration with Kreszig (1989) integral equation of form

$$x(t) = \mu \int_a^b K(t, \tau) x(\tau) x(\tau) d\tau + v(t) \qquad 2.19$$

Which is a Fredholm equation of the second kind with the Kernel K a given function on the square G = [a,b] x [a,b].
Let us consider (3.19) in the space of continuous function C[a,b] and defined on the interval.
J = [a,b] with the norm

$\|x - y\| = \max \|x(t) - y(t)\|, \quad t \in J$ 2.20

Assuming that C [a,b] is complete and v ϵ C [a,b] is continuous on G, then K is a bounded function on G such that

$K_{max} \leq M.$ 2.21

Rewriting (2.19) as a fixed – point problem, we have
x = F (x) and

$$F(x(t) = v(t) + \mu \int_a^b K(t, \tau)\, x(\tau) d\tau \qquad 2.22$$

Equation (2.2.2) defines an operator F: C [a,b] $\longrightarrow$ c[a,b].

Imposing a restriction on μ so that F becomes a contraction, we have from (2.20) to (2.22)

$$\| F(x) - F(y) \| \leq \max \| F((x(t)) - F(y(t)) \|, \quad t \epsilon J$$

$$\leq |\mu| \max \left| \int_a^b K(t,\tau)\,[x(\tau) - y(\tau)] \right| d\tau \leq |\mu| \max \int_a^b |k(t, \tau)|\, |x(\tau) - y(\tau)|\, d\tau$$

$$\leq |\mu| M \|x - y\| (b - a).$$

Choosing $\alpha = |\mu| M (b - a)$,

$\| F(x) - F(y) \| \leq \alpha \| x - y \|$. Then F is a contraction (α <1) if $|\mu| < \dfrac{1}{M(b-a)}$

It does follow that F is a contraction mapping of $\overline{B}(x_0, r_0)$ with $\alpha = |\mu| M(b - a)$, for some x_0 and

$$r_0 = \frac{1}{1-a} \|x_1 - x_0\| \leq \theta \leq r$$

where θ is a real number.
Hence, theorem (2.1) guarantees the existence of a fixed point x^* of F in $\overline{B}(x_0, \theta)$
to which the sequence $\{x_m\}$ defined by

$$x_{m+1}(t) = v(t) + \mu \int_a^b K(t,\tau) x_m(\tau) d\tau$$

Converges with

$\|x^* - x_m\| \leq \alpha^m \theta$ 2.24

Example
We apply the above illustration to the operator equation given by

$F(x)(s) = F(s) + \lambda \int_0^1 K(s,t)\, x(t)\, dt$ in the ball $\overline{B}(1,½)$,

Where

$$K(s.t) = \begin{cases} s(1-t), s \leq t \\ t(1-s), t \leq s \end{cases}$$

Let $M = \max_{s\,[0,1]} \left| \int_0^1 K(s,t)\, dt \right|$

then

$\|F(x) - F(y)\| \leq |\lambda| M \|x - y\| \;\; \frac{\lambda}{8} \;\; \lambda \;\; \|x - y\|.$

For $|\lambda| \leq 1$

$\|F(x) - F(y)\| \leq 0.125 \|x - y\|$

giving $\alpha = 0.125$. and for $x_0 = f(s) = 1$

$$r_0 = \frac{1}{1 - 0.125}\, 0.125 \leq 0.143 < r,$$

So that there exists a fixed point x^* of F in $\bar{B}$ (1,0,) to which the sequence $\{x_m)$ defined by

$$x_{m+1}(s) = f(s) + \lambda \int_0^1 k(s,t)\, x_m(t)\, dt, \; m = 0,1,2,3, \qquad 2.25$$

Converges with

$\|x^* - x_m\| \leq 0.125^m\, 0.143.$

3 Iterative Estimate of Solutions of Nonlinear Integral Equations

Introduction

This section is concerned with the application of iteration methods discussed in the previous section to establish the location and existence of approximate solutions of some nonlinear integral equations of Fredholm type and second kind. In addition, we will obtain sequences of iterates.

Application of the Contraction Mapping Principle to Nonlinear Integral Equation of Rediative Transfer
The ideas of section 2 will be applied to the nonlinear equation

$$x(s) = \lambda\, x(s) \int_0^1 \frac{s x(t)}{s+1}\, dt + 1, \; s \in [0,1]] \text{ , and } \lambda > 0 \qquad 3.1$$

Known as Chandrasekhar's integral equation, Ahues (2004). The equation of the type (3.1) is associated with the equation arising from the theory of radiative transfer.

Our intention will be to obtain the locations, the existence and uniqueness of the solution of (4.1.1) in the ball B (x_0, r)

We note that equation (3.1) suggests the direct iteration.

$$x_{m+1}(S) = \lambda x_m(s) s \int_0^1 \frac{x_m(t)}{s+1}\, dt + 1$$

With $x_0(s) = 1$, $0 \leq s \leq 1$

This indicates that

$$x_1(s) = \lambda\, s \ln(1 + 1/s) + 1, 0 \leq s \leq 1 \qquad 3.3$$

the exact calculate of x_2 (s) however involves the evaluation of the integral

$$\int_0^1 \frac{s}{s+1} In(1+1/s)\,dt, s \in [0,1] \qquad 3.4$$

Which leads to analytic complexities. However by (3.3) and (3.4) , we have,

$\|x_1 - x_0\|$ = max $\|x_1(s) - x_0(s)\|$ = λ1nn2 3.5

Or

$\|x_1 - x_0\| < 0.6931\ \lambda.$ 3.6

Expressing (3.1) as a fixed – point problem in C [0,1] , we obtain

$$F(x)(s) = \lambda x(s) \int_0^1 \frac{sx(t)}{s+1} dt + 1 \qquad 3.7$$

which is a contraction mapping of the ball $\overline{B}$ (1, r) with the contraction factor α, and

$$r \geq \frac{1}{1-a} \lambda\ In2. \qquad 3.8$$

The existence of the solution x^* (s) of (3.1) and the convergence of the sequence x_m generated by (3.2) to it follows from contraction mapping principle (Theorem 2.1).

To determine the value of α , note that.

$$F(x) - F(y) = \lambda s[x(s)\int_0^1 \frac{x(t)}{s+1} dt - y(s)\int_0^1 \frac{y(t)}{s+1} dt]$$

$$= \tfrac{1}{2} \lambda_s \{[x(s)+y(s)]\int_0^1 \frac{1}{s+1}[x(t)-y(t)]dt + [x(s)-y(s)]\int_0^1 \frac{1}{s+1}[x(t)-y(t)]dt\} \qquad 3.9$$

If x,y ϵ $\overline{B}$ (1,r), then

$\|x + y\| \leq 2\ (1 + r)$ 3.10

And

Max x(t) – y(t)|dt| ≤ 2(1 + r)1n2 3.11

It follows from (3.11) that

$\|F(x) - F(y)\| \leq 2\ \lambda\ (1 + r)$ 1n2 $\|x - y\|$ 3.12

And F is a contraction mapping of the ball $\overline{B}$ (1,r) if

$\alpha = 2\ \lambda(1 + r)\ 1n2 < 1$ 3.13

From (3.10) and (3.15) the hypothesis of Theorem 2.1 will be satisfied for a given

Value of λ if r $\geq \dfrac{1-2p}{4p} + \sqrt{\left(\dfrac{1-2p}{4p}\right)^2 - \frac{1}{2}}$ 3.14

Holds for

$P = \lambda 1n2$ 3.15

This means

$2pr^2 - (1 - 2p)\ r + p \leq 0$ 3.16

Which has a formal solution

$$\frac{1-2p-\sqrt{((1-2p)^2-8p^2)}}{4p} \leq r \geq \frac{1-2p+\sqrt{((1-2p)^2-8p^2)}}{4p}$$

If we assume that $(1 - 2p)^2 - 8p^2 \geq 0$, 3.18

Then, the maximum value of p for which this will be possible is determined from $(1 - 2p)^2 - 8p^2 = 0$. this will now give the value of p as

$$P = \frac{\sqrt{2-1}}{2} \qquad 3.19$$

And for this value of p, (4.1.19) gives

$$r = \frac{\sqrt{2}}{2} \qquad 3.20$$

From (3.1) and (3.22)

$\alpha = 2p(1 + r) <1$, .3.21

So that (3.15) is satisfied . Thus x_0 (s) = 1, s ϵ [0,1) is a satisfactory initial approximation to the solution of (3.1) if

$$0 < \lambda \leq \frac{\sqrt{2-1}}{2In2} = 0.29879 \qquad 3.22$$

For each λ in this range, Theorem 2.1 guarantees the existence of a unique solution in the ball $\overline{B}$ $(1,r_0)$. where

i.e a unique solution exist in the ball $\overline{B}(1,0.7071\ldots)$

Remark 4.1.1

Due to analytic complexities involved after the first iterate of (3.10) subsequent results can be obtained by the use of methods of numerical integration.

References

Ahues , M, 2004. Newton methods with Holder derivative, Numerical Functional Analysis and Optimization, 25(5-6): 379-395.

Atman, M., 1977., Contractor and contractor directions, Theory and Applications, Marcel Dekker, New York.

Argyros, I.K., 1985. Quadratic equations and applications to Chandrasekhar's and related equations, Bull. Australian Math. Soc., 32: 275-297.

Argyros, I.K., 2005. Approximate Solution of Operation Equations with Applications, World Scientific Publishing Co. Hachersack, New Jersey USA.

Atkinson, K.E., 1992. A survey of numerical methods for solving nonlinear integral.