An Internal Penny-Shaped Crack Problem in an Infinite Thermoelastic Solid

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Abstract: A problem for an infinite thermoelastic solid weakened by an internal penny-shaped crack has been solved. The solid that is assumed to be homogeneous and isotropic is subjected to temperature and stress distributions. A cylindrical system of coordinates is used, in which the plane \( z = 0 \) is that of the crack and the \( z \)-axis is normal to it at the centre. In addition, the crack occupies the region \( z = 0, 0 \leq r \leq a \) that is subjected to prescribed temperature and stress distributions which vary with the radial distance \( r \). The problem is solved using the Hankel transform. The boundary conditions of the problem give a set of two dual integral equations, which are solved analytically. The inversion of the transform is then obtained analytically. Numerical results for the temperature, stress and displacements distributions are shown graphically and then discussed. All the definite integrals involved were calculated using Romberg technique of numerical integration with the aid of a Fortran program compiled with Visual Fortran v.6.1 on a Pentium-IV pc with processor speed 2.0 GHz.

Key words:

INTRODUCTION

According to the theory of generalized thermoelasticity with one relaxation time, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law of heat conduction\[1\]. The heat equation associated with this theory is hyperbolic, eliminating automatically the paradox of infinite speeds of propagation inherent in previous uncoupled and the coupled theories of thermoelasticity. Such a theory was extended to include both the effects of anisotropy and the presence of heat sources\[2\].

There have been several investigations dealing with problems of various aspects of the aforementioned theory. Among the theoretical contributions by subject are proofs of uniqueness theorems under different conditions\[2-5\]. The state space approach to this theory was developed for one-dimensional problems\[6,7\] as well as for two-dimensional problems\[9\]. In addition to the fundamental solution for this problem, two-dimensional problem for a thick layer and a dynamical problem for mode-I crack in an infinite space have been solved\[8-11\].

In recent years, considerable effort has been devoted to the study of cracks in solids, due to their applications in industry in general and in the fabrication of electronic components in particular. They have occurred for many reasons, including uncertainties in the loading or environment, defects in materials, inadequacies in design and deficiencies in construction or maintenance. Consequently, all structures contain cracks, as manufacturing defects or because of service loading, which can be either mechanical or thermal. If the load is frequently applied, the crack may grow in fatigue to a final fracture. As the size of the crack increases, the residual strength of the structure ceases. In the final stages of the crack growth, the rate increases suddenly leading to a catastrophically structure failure.

Study of such failure mechanics helps to maintain the structural integrity due to cracks. In 1983, the National Institute for Science and Technology and Battelle Memorial Institute estimated high costs for failure due to fracture\[10\]. A similar study commissioned by the European Union concluded that billions of ECU per year could be saved using fracture mechanics technology.

MATERIALS AND METHODS

Formulation of the problem: The cylindrical system of coordinates will be used, in which the plane \( z = 0 \) is the plane of the crack and the \( z \)-axis is normal to the crack at
its centre. The crack occupies the region \( z = 0, 0 \leq r \leq a \) and is subjected to prescribed distributions that vary with the radial distance \( r \), where \( a \) is the radius of the crack, as shown in figure (1). The solid is assumed to be homogeneous, isotropic and elastic. Since the geometry of the region is symmetric about the crack plane, the problem is reduced to a mixed boundary value problem of theroelasticity for the region \( z \geq 0, r \geq 0 \). All considered functions will depend on \( r \) and \( z \) only.

\[
e = \frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}
\]

\[
= \frac{1}{r} \frac{\partial}{\partial r} \left( ru \right) + \frac{\partial w}{\partial z}
\] (4)

\( \nabla^2 \) is the two-dimensional Laplacian operator in a cylindrical coordinates system that takes the form:

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}
\]

The stress components, \( \mathbf{F} \), expressed by the following constitutive relations that supplement the above equations:

\[
\sigma_r = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma (T - T_0)
\] (5a)

\[
\sigma_z = 2\mu \frac{\partial w}{\partial z} + \lambda e - \gamma (T - T_0)
\] (5b)

\[
\sigma_r = \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)
\] (5c)

Making use of the following non-dimensional variables:

\[
\begin{aligned}
r' &= c_i \eta r, & z' &= c_i \eta z, & u' &= c_i \eta u, \\
w' &= c_i \eta w, & \sigma_y' &= \frac{\sigma_y}{\mu} & \text{and} & \theta &= \frac{\gamma (T - T_0)}{(\lambda + 2\mu)},
\end{aligned}
\]

where \( \eta = \frac{\rho c_p}{k} \), \( c_i \) is the speed of propagation of isothermal elastic waves given by: \( c_i = \sqrt{\frac{\lambda + 2\mu}{\rho}} \), in which \( D \) is the density and \( k \) is the material’s thermal conductivity.

Using the above non-dimensional variables, the governing equations take the following form (dropping the primes for convenience):

\[
\nabla^2 u - \frac{u}{r^2} + \left( \beta^2 - 1 \right) \frac{\partial e}{\partial r} - \beta^2 \frac{\partial \theta}{\partial r} = 0
\] (6)

\[
\nabla^2 w + \left( \beta^2 - 1 \right) \frac{\partial e}{\partial z} - \beta^2 \frac{\partial \theta}{\partial z} = 0
\] (7)

\[
\nabla \theta = 0
\] (8)

while the stress components (5a)-(5c) are reformulated to become:
\[ \sigma_r = 2 \frac{\partial v}{\partial r} + \left( \beta^2 - 2 \right) e - \beta^3 \theta \]  
(9.a)

\[ \sigma_{zz} = 2 \frac{\partial w}{\partial z} + \left( \beta^2 - 2 \right) e - \beta^3 \theta \]  
(9.b)

\[ \sigma_{r} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial r} \]  
(9.c)

In the above equations, note that \( \beta^2 = \frac{\lambda + 2\mu}{\mu} \).

Combining equations (6) and (7), regarding equations (4) and (8) to get:

\[ \nabla^2 e = 0 \]  
(10)

The boundary conditions for the problem at \( z = 0 \) may be taken as:

\[ \theta (r, 0) = f(r), \quad 0 < r < a \]  
(11.a)

\[ \sigma_{zz} (r, 0) = -p(r), \quad 0 < r < a \]  
(11.b)

\[ \frac{\partial \theta}{\partial z} (r, 0) = 0, \quad a < r < \infty \]  
(11.c)

\[ w(r, 0) = 0, \quad a < r < \infty \]  
(11.d)

\[ \sigma_{r} (r, 0) = 0, \quad 0 < r < \infty \]  
(11.e)

where \( f(r) \) and \( p(r) \) are known functions of temperature and mechanical stress, respectively.

**Analytical solution of the problem:** The Hankel transform with parameter \( \alpha \) of a function \( f(r, z) \) denoted by \( f^*(\alpha, z) \) is given by the relation \(^{17}\):

\[ f^*(\alpha, z) = H \left[ f(r, z) \right] = \int_{0}^{\infty} f(r, z) J_0(\alpha r) \, dr \]

where \( J_n(h) \) is the Bessel function of the first kind of order \( n \).

The inverse Hankel transform is given by the relation \(^{16,17}\):

\[ f(r, z) = H^{-1} \left[ f^*(\alpha, z) \right] = \int_{0}^{\infty} f^*(\alpha, z) \alpha J_0(\alpha r) \, d\alpha \]

Taking the Hankel transform with parameter \( \alpha \) of both sides of equation (8) and using the following operational relation of the Hankel transform\(^{17}\):

\[ \left( D^2 - \alpha^2 \right) w^* = 0 \]  
where \( D = \partial / \partial z \)

The solution of the above equation, which is bounded at infinity, can be written as

\[ \theta^*(\alpha, z) = A(\alpha) e^{-\alpha z} \]

where \( A(\alpha) \) is a parameter depending on \( \alpha \) only.

Due to symmetry, only the case where \( z > 0 \) will be considered, accordingly:

\[ \theta^*(\alpha, z) = A(\alpha) e^{-\alpha z}, \]

(12)

Taking the inverse Hankel transform of both sides of equation (12), gives:

\[ \theta(r, z) = \int_{0}^{\infty} A(\alpha) e^{-\alpha z} \alpha J_0(\alpha r) \, d\alpha, \]

(13)

Similarly, since \( e \) satisfies the same differential equation as \( \theta \), \( e^* \) can be written in the form

\[ e^*(\alpha, z) = \frac{\beta^2 A(\alpha) - 2B(\alpha)}{\beta^2 - 1} e^{-\alpha z}, \]

(14)

where \( B(\alpha) \) is a parameter depending on \( \alpha \) only.

Applying the inverse Hankel transform for equation (14), it reads

\[ e(r, z) = \int_{0}^{\infty} \frac{\beta^2 A(\alpha) - 2B(\alpha)}{\beta^2 - 1} e^{-\alpha z} \alpha J_0(\alpha r) \, d\alpha, \]

(15)

Regarding equations (12) and (14) and applying the Hankel transform to both sides of equation (7), the latter became:

\[ \left( D^2 - \alpha^2 \right) w^* = -2\alpha B(\alpha) e^{-\alpha z} \]

(16)

Equation (16) has the following solution, for \( z > 0 \), which is bounded at infinity is given by

\[ w^*(\alpha, z) = \left[ C(\alpha) + B(\alpha) z \right] e^{-\alpha z}, \]

(17)

where \( C(\alpha) \) is a parameter depending on \( \alpha \) only.
Inverting the Hankel transform, equation (17) tends to:

\[ w(r, z) = \int_0^\infty \left[ C(\alpha) + B(\alpha) z \right] e^{-\alpha z} \alpha J_0(\alpha r) d\alpha, \quad (18) \]

Considering equations (14) and (17) and taking the Hankel transform of both sides of equation (4),

\[ H \left[ \frac{1}{r} \frac{\partial}{\partial r} (ru) \right] = \left[ \frac{\beta^2 A(\alpha) - (\beta^2 + 1) B(\alpha)}{\beta^2 - 1} + \alpha C(\alpha) + \alpha B(\alpha) \right] e^{-\alpha z} \quad (19) \]

It is not difficult to show that the solution of equation (19) is given by

\[ u(r, z) = \int_0^\infty \left[ \frac{\beta^2 A(\alpha) - (\beta^2 + 1) B(\alpha)}{\beta^2 - 1} + \alpha C(\alpha) + \alpha B(\alpha) \right] e^{-\alpha z} J_1(\alpha r) d\alpha \quad (20) \]

Substituting from equations (13), (15), (18) and (20) into equations (9.b) and (9.c) and using the following formula\(^{16}\):

\[ \frac{dJ_0(z)}{dz} = -J_1(z) \quad (21) \]

the stress tensor components will take the form:

\[ \sigma_{zz} = -\int_0^\infty \left[ \frac{\beta^2 A(\alpha) - 2B(\alpha)}{\beta^2 - 1} + 2\alpha C(\alpha) + 2\alpha B(\alpha) \right] e^{-\alpha z} \alpha J_0(\alpha r) d\alpha \quad (22) \]

\[ \sigma_{rz} = -\int_0^\infty \left[ \frac{\beta^2 \left[ A(\alpha) - 2B(\alpha) \right]}{\beta^2 - 1} + 2\alpha C(\alpha) + 2\alpha B(\alpha) \right] \alpha e^{-\alpha z} J_1(\alpha r) d\alpha \quad (23) \]

Substituting from equations (13), (18), (20), (22) and (23) into the boundary conditions (11), the following relations are obtained:

\[ \int_0^\infty A(\alpha) \alpha J_0(\alpha r) d\alpha = f(r), \quad 0 < r < a \quad (24) \]

\[ \int_0^\infty \left[ \frac{\beta^2 A(\alpha) - 2B(\alpha)}{\beta^2 - 1} + 2\alpha C(\alpha) \right] \alpha J_0(\alpha r) d\alpha = \frac{p(r)}{2}, \quad 0 < r < a \quad (25) \]

\[ \int_0^\infty \left[ \frac{\beta^2 \left[ A(\alpha) - 2B(\alpha) \right]}{\beta^2 - 1} + 2\alpha C(\alpha) \right] \alpha J_1(\alpha r) d\alpha = 0, \quad 0 < r < \infty \quad (26) \]

\[ \int_0^\infty A(\alpha) \alpha^2 J_0(\alpha) d\alpha = 0, \quad a < r < \infty \quad (27) \]

\[ \int_0^\infty C(\alpha) \alpha J_0(\alpha r) d\alpha = 0, \quad a < r < \infty \quad (28) \]
Since equation (26) is valid for all values of \( r \), \( B''(\alpha) \) is obtained in the form
\[
B'(\alpha) = \frac{\beta^2 A'(\alpha) + 2\alpha \left( \beta^2 - 1 \right) C'(\alpha)}{2\beta^2} \quad (29)
\]

Substituting for \( B''(\alpha) \) from equation (29) and using equation (24), equation (25) reduces to:
\[
\int_0^\infty C'(\alpha) \alpha^2 J_0(\alpha r) d(\alpha) = \frac{\beta^2}{2 \left( \beta^2 - 1 \right)} \int_0^\infty \left[ p(r) - f(r) \right] d(\alpha), \quad 0 < r < a \quad (30)
\]

Equations (24) and (27) are a set of a dual integral equations whose solution gives the unknown variable \( A''(\alpha) \), also equations (28) and (30) are a set of a dual integral equations, the solution of which gives the unknown variable \( C''(\alpha) \). The solution of the dual integral equations (24) and (27) is given by \(^{19}\):
\[
A'(\alpha) = \frac{2}{\pi \alpha} \int_0^\alpha \cos(\alpha u) \left\{ \frac{d}{du} \int_0^u r f(r) dr \right\} du \quad (31)
\]

The solution of the dual integral equations (28) and (30) has the form \(^{19}\):
\[
C(\alpha) = \frac{\beta^2}{\pi \alpha \left( \beta^2 - 1 \right)} \int_0^\alpha t \sin(\alpha t) \left\{ \int_0^{\pi / 2} \sin \theta \left[ p(t \sin \theta) - f(t \sin \theta) \right] d\theta \right\} dt \quad (32)
\]

**Numerical analyses:** In what follows we shall take
\[
f(r) = 1, \quad 0 < r < a \quad \quad p(r) = r^2, \quad 0 < r < a
\]

Substituting these values into equations (31) and (32), and after some manipulations,
\[
A'(\alpha) = \frac{2 \sin \alpha a}{\pi \alpha^2} \quad (33)
\]

\[
C(\alpha) = \frac{\beta^2}{\pi \alpha \left( \beta^2 - 1 \right)} \left[ \frac{3a - 2a^3}{3 \alpha} \cos(\alpha a) + \frac{2a^2 - 1}{\alpha^2} \sin(\alpha a) + \frac{4a \cos(\alpha a)}{\alpha^3} - \frac{4 \sin(\alpha a)}{\alpha^4} \right] \quad (34)
\]

Substituting from equations (33) and (34) into relation (29),
\[
B'(\alpha) = \frac{1}{\pi} \left[ \frac{3a - 2a^3}{3 \alpha} \cos(\alpha a) + \frac{2a^2}{\alpha^2} \sin(\alpha a) + \frac{4a \cos(\alpha a)}{\alpha^3} - \frac{4 \sin(\alpha a)}{\alpha^4} \right] \quad (35)
\]

Substituting from equation (33)-(35) into equation (22),
\[
\sigma_\alpha(r, z) = -\frac{2}{\pi} \left[ \frac{3a - 2a^3}{3} \cos(\alpha a) + \frac{2a^2}{\alpha} \sin(\alpha a) + \frac{4a \cos(\alpha a)}{\alpha^2} - \frac{4 \sin(\alpha a)}{\alpha^3} \right] (1 + \alpha z) e^{-\alpha J_0(\alpha r)} d\alpha \quad (36)
\]
Putting \( z = 0 \),

\[
\sigma_{zz} (r, 0) = -\frac{2}{\pi} \int_0^\infty \left[ \frac{3a - 2a^3}{3\alpha} \cos (\alpha a) + \frac{2a^2 \sin (\alpha a)}{\alpha^2} + \frac{4a \cos (\alpha a)}{\alpha^3} - \frac{4 \sin (\alpha a)}{\alpha^4} \right] J_0 (\alpha r) \, d\alpha
\]  

(37)

In order to evaluate the integral on the right hand side of equation (37) for \( r > a \), the following two integral formulae of the Bessel functions \([18],[20]\) are to be used:

\[
\int_0^\infty \frac{\sin \alpha b J_0 (\alpha r) \, d\alpha}{\alpha} = \begin{cases} 
\frac{\pi}{2}, & r < b \\
\sin^{-1} \frac{b}{r}, & r > b
\end{cases}
\]  

(38)

\[
\int_0^\infty \cos (\alpha b) J_0 (\alpha r) \, d\alpha = \begin{cases} 
0, & r < b \\
\frac{1}{\sqrt{r^2 - b^2}}, & r > b
\end{cases}
\]  

(39)

Multiplying both sides of the relation (38) by \( b \) and integrating the resulting relation with respect to \( b \) over the range \((0,a)\), to obtain for \( r > a \):

\[
\int_0^a \left[ \frac{\sin (\alpha a) - a \cos (\alpha a)}{\alpha^3} \right] J_0 (\alpha r) \, d\alpha = \frac{a^2}{2} \sin^{-1} \left( \frac{a}{r} \right) - \frac{r^2}{4} \left[ \sin^{-1} \left( \frac{a}{r} \right) - \frac{a \sqrt{r^2 - a^2}}{r^2} \right]
\]  

(40)

Using the relations (38)-(40), equation (37) becomes

\[
\sigma_{zz} (r, 0) = -\frac{2}{\pi} \left[ \sin^{-1} \left( \frac{a}{r} \right) + \frac{a (3 + a^2 - 3r^2)}{\sqrt{r^2 - a^2}} \right], \quad r > a
\]  

(41)

Substituting from equation (33) into equation (13),

\[
\theta (r, z) = \frac{2}{\pi} \int_0^\infty \frac{\sin (\alpha a)}{\alpha} e^{-\alpha z} J_0 (\alpha r) \, d\alpha
\]  

(42)

for \( z = 0 \),

\[
\theta (r, 0) = \frac{2}{\pi} \int_0^\infty \frac{\sin (\alpha a)}{\alpha} J_0 (\alpha r) \, d\alpha,
\]  

(43)

Using the integral formula (38), equation (43) becomes:

\[
\theta (r, 0) = \frac{2}{\pi} \sin^{-1} \frac{a}{r}, \quad \text{for } r > a
\]  

(44)

Substituting from equations (33)-(35) into equation (20),

\[
u (r, 0) = -\frac{2}{\pi} \int_0^\infty \left[ \frac{(3a - 2a^3)}{3\alpha} \cos (\alpha a) + \frac{(2a^2 - \beta^2)}{\alpha^2} \sin (\alpha a) + \frac{4a \cos (\alpha a)}{\alpha^3} - \frac{4 \sin (\alpha a)}{\alpha^4} \right] J_0 (\alpha r) \, d\alpha
\]  

(45)
In a similar way, to evaluate the integral on the right hand side of the above equation, when \( r > a \) the two integral formulas of the Bessel functions\(^{[18],[20]}\) are to be used, where:

\[
\int_0^\infty \sin(\alpha b) J_1(\alpha b) = \begin{cases} 
0, & r < b \\
\frac{b}{r \sqrt{r^2 - b^2}}, & r > b 
\end{cases} 
\]

(46)

\[
\int_0^\infty \cos(\alpha b) \frac{J_1(\alpha b)}{\alpha} = \begin{cases} 
0, & r < b \\
\frac{r^2 - b^2}{r}, & r > b 
\end{cases} 
\]

(47)

Integrating equation (45), after multiplying both sides by \( b^2 \), with respect to \( b \) over the range \((0,a)\) gives:

\[
\int_0^a \left[ \frac{-2a^3 \cos(\alpha a)}{3\alpha} + \frac{2a^2 \sin(\alpha a)}{\alpha^2} + \frac{4a \cos(\alpha a)}{\alpha^3} - \frac{4 \sin(\alpha a)}{\alpha^4} \right] J_1(\alpha r) d\alpha 
\]

\[
= \frac{r^3}{6} \left[ \frac{3}{2} \sin^{-1} \left( \frac{a}{r} \right) - 2a \sqrt{r^2 - a^2} \right] + \frac{a \sqrt{r^2 - a^2}}{2r^2} \left( r^2 - 2a^2 \right), \quad \text{for } r > a,
\]

(48)

Also integrating both sides of the relation (46) with respect to \( b \) over the range \((0,a)\) leads to:

\[
\int_0^a \sin(\alpha a) \frac{J_1(\alpha r)}{\alpha^2} d\alpha = \frac{r}{2} \left( \sin^{-1} \left( \frac{a}{r} \right) - \frac{a \sqrt{r^2 - a^2}}{r^2} \right), \quad r > a
\]

(49)

Substituting from equations (46)-(49) into equation (45) to get

\[
u(r,0) = \frac{-1}{\pi (\beta^2 - 1)} \left\{ \frac{r(r^2 - 2\beta^2)}{4} \sin^{-1} \left( \frac{a}{r} \right) + \frac{12 + 6\beta^2 + r^2(2 - 2a^2 - 4) \sqrt{r^2 - a^2}}{12r} \right\}, \quad r > a
\]

(50)

**RESULTS AND DISCUSSIONS**

The above evaluations are applied to copper material, whose constants are shown in table 1.

The computations were performed for different values of \( z \) as shown in figures(2-6). All the definite integrals involved were calculated using Romberg technique of numerical integration with variable step size, upon using a computer program compiled with Visual Fortran v.6.1 on a Pentium-IV pc with processor speed 2.0 GHz.

Figure (2) displays the distribution of the temperature, \( 2 \), versus the radial distance, \( r \), at various values of the axial distance, \( z \). Note that the crack’s radius, \( r \), is unity or is taken to be the unit of length in the problem so that \( 0 \leq r \leq 1 \), \( z=0 \), and that the crack is symmetric with respect to the \( z\)-plane. It is clear from the graph that \( 2 \) has its maximum value at the centre of the crack, it begin to fall just near the crack edge, where it experiences sharp decreases (with maximum negative gradient at the crack’s circumference). Graph lines for different values of \( z \) show different slopes at crack ends according to \( z\)-values. In other words, the temperature line for \( z=0.2 \) has the highest gradient when compared with that of \( z=0.4, 0.6 \) then \( 0.8 \), at the edge of the crack. Besides, all lines begin to coincide when the radial distance \( r \) is beyond the double of the crack radius to reach the reference temperature of the solid. These results obey physical reality for the behaviour of copper as a polycrystalline solid.

Concerning plots for the radial and axial displacements, \( u \) and \( w \), shown in figures (3) and (4), respectively, it can be concluded that a change of volume is attended by a change of the temperature while the effect of the deformation upon the temperature distribution is the subject of the theory of thermoelasticity. The solid particles radial displacement, \( u \), shows an increase to reach its maximum magnitude just after the crack circumference and beyond it \( u \) falls again to try to attain zero at infinity. It is clear that maximum displacements are strongly dependent on the axial distance, \( z \). Moreover, \( u \) rises at a decreasing rate with
Table 1: Thermal and elastic constants for copper.

\[
\begin{array}{|c|c|c|}
\hline
\text{Parameter} & \text{Value} & \text{Value} \\
\hline
n & 1.78 \times 10^9 \text{K}^3 & c_l = 383.1 \text{ J/kgK} \\
\hline
\rho & 8886.73 \text{ m/s}^2 & c_t = 4.158 \times 10^3 \text{ m/s} \\
\hline
\mu & 8886.73 \text{ m/s} & \gamma = 7.76 \times 10^5 \text{ N/m}^2 \\
\hline
\beta & 8954 \text{ kg/m}^2 & \rho = 4 \text{ T} \\
\hline
T_0 & 293 \text{ K} & a = 1 \\
\hline
\end{array}
\]

Figure (2) Temperature Distribution

Figure (3) Radial Displacement Distribution

Figure (4) Axial Displacement Distribution
Nomenclature

\( T \) absolute temperature
\( \alpha \) coefficient of linear thermal expansion.
\( U \) component of the horizontal displacement
\( V \) component of the vertical displacement
\( F_{zz} \) components of stress tensor in the axial direction
\( F_{rr} \) components of stress tensor in the radial direction
\( D \) density
\( E \) dilatation
\( \mathbf{U} \) displacement vector
\( H[ f(r,z) ] \) Hankel transform
\( h^{-1}[ f(r,z) ] \) Hankel’s inverse transform
\( c_1 \) is the speed of propagation of isothermal elastic waves
\( p(r) \) known function of mechanical stress
\( f(r) \) known function of temperature
\( \delta \) and \( \mu \) Lamé’s elastic constants: \( \lambda = \frac{2\mu\nu}{1-2\nu} \) and \( \mu = \frac{E}{2(1+\nu)} \)

\( \nabla^2 \) Laplacian operator
\( K \) Material’s thermal conductivity
\( < \) Poisson’s ratio
\( T_o \) Reference temperature
\( \mu \) Shear’s modulus
\( c_E \) specific heat at constant strain
2 Temperature change above reference temperature

\( E \) Young’s modulus

increasing \( z \), when we go vertically farther from the crack. On the other hand, figure (4) shows that the axial displacement, \( w \), increases to its maximum magnitude just at the crack edge. While beyond this edge, \( w \) decreases to reach zero (state of particles equilibrium) at the radial distance that is four times that of the crack radius (\( a=1 \)). The displacements \( u \) and \( w \) show different behaviours, because of the elasticity of the solid that tends to resist vertical displacements in the studied problem.

Figures (5) and (6) display radial and axial stress component distributions, \( F_{rr} \) and \( F_{zz} \), respectively. Both of them has its maximum amplitude just at the crack edge, but
F_r reaches zero at radial distance equal to only three times that of the crack radius whereas, F_r reaches zero at infinity. Variation of z has a serious effect on the magnitudes of both mechanical stresses. Such effect on the radial stress is in opposite manner to that on the axial one. Since all functions are continuous, especially that of u and w, it proves that the crack will not propagate because of the fact that both of the mechanical and thermal stresses are not sufficient to propagate such a crack, and to propagate it, the solid need to be subjected to an external stress (tensile, shear, ...).

Conclusions:

1. Analytical solutions based upon Hankel transformation for thermoelastic problem in solids have been developed and utilized.
2. Implement such solutions for a penny-shaped crack in a composite solid.
3. Temperature, radial and axial distributions were estimated at different distances from the centre of the crack.
4. radial and axial stress distributions were evaluated as functions of the distance from the crack centre.
5. Crack dimensions are significant to elucidate the mechanical structure of the solid.
6. cracks are stationary and external stress is demanded to propagate such cracks.

REFERENCES