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Comparison of Single Term Haar Wavelet Series Technique and Euler method to Solve One Dimensional Fuzzy Differential Inclusions

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ABSTRACT

This paper presents a comparison of single-term Haar wavelet series (STHW) technique and the Euler method to solve one dimensional fuzzy differential inclusions (E.Babolian, S.Abbasbandy and M.Alavi 2009). Fuzzy reachable set can be approximated by STHW technique with complete analysis. The discrete solutions obtained through STHW technique are compared with that of the Euler method. The applicability of the STHW technique is more suitable to solve one dimensional fuzzy differential inclusions.

Key words: Haar wavelet; single-term Haar wavelet series (STHW), Differential inclusions, Fuzzy sets, Fuzzy differential inclusions.

Introduction

The reachable set of a differential inclusion (the latter interpreted as a uncertain system) is the minimal guaranteed estimation of the current state. Therefore, to calculate reachable sets is a cornerstone of the estimation and control of uncertain systems (Kurzhanski, B. and T.F. Filippova, 1993). A lot of work has been done for developing numerical approximation methods, see the surveys (Dontchev, A. and F. Lempio, 1992; Lempio, F. and V.M. Veliov, 1998). Since the geometry of the reachable sets could be rather complicated, specific subclasses of sets are usually used as approximation tools: boxes, polyhedral sets, ellipsoids (Chernousko, F.L., 1988; Kurzhanski, A.B. and I. V’alyi, 1997; Chernousko, F.L and D.Ya. Rokityanskii, 2000; Kurzhanski, A.B. and P. Varaiya, 2000), box or polyhedral complexes (Saint-Pierre, P., 1994; H’ackl, G., 1992-93; Ushakov, V. and A. Khrupinov, 1994; Cardaliaguet, P. et al., 1999), etc. In some cases convergence results are obtained, but usually to achieve a good approximation one has to use rather complex approximating sets.

STHW plays an important role in both the analysis and numerical solution of differential inclusions. STHW can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. However, working with fuzzy differential equations places special demands on STHW codes. In science and engineering, fuzzy differential inclusions often have to be solved. Although some cases can be solved analytically, the majority of fuzzy differential inclusions are too complicated to have analytical solutions. Even when analytical solutions can be found, they are not always useful in practice since the computational cost involved is very high.

In recent years, there has been an increased interest in several methods were arisen to solve the fuzzy differential inclusions. STHW can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and team of his researchers (Sekar, S., and A. Manommai, 2009; Sekar, S., and G. Balaji, 2010; Sekar, S., and M. Duraisamy, 2010; Sekar, S., and K. Jaganathan, 2010; Sekar, S., and R. Kumar, 2011; Sekar, S., and E. Paramanathan, 2011; Sekar, S., and M. Vijayarakavan, 2010) introduced the STHW to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere, to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations, to study on periodic and oscillatory problems, and numerical solution of nonlinear problems in the calculus of variations. In this paper, we consider the one dimensional fuzzy differential inclusions in E.Babolian, S.Abbasbandy and M.Alavi (2009) and solve by using the STHW technique. The results are compared with Euler method E.Babolian, S.Abbasbandy and M.Alavi (2009).
Preliminaries of fuzzy differential inclusions:

Prior to introduce fuzzy differential inclusion we must denote fuzzy sets and fuzzy numbers as follows. We place a tilde over a symbol to denote a fuzzy set so \( \tilde{X}, \tilde{A}, \ldots \) all represent fuzzy subsets in \( \mathbb{R} \). We write \( tX \) for the membership function of \( \tilde{X} \) evaluated at \( t \in \mathbb{R} \). An \( \alpha \)-cut of \( \tilde{X} \) written \( \tilde{X}_\alpha \), is defined as \( \{ f : \tilde{X}(f) \geq \alpha \} \), for \( 0 < \alpha < 1 \) and \( \tilde{X} = \bigcup_{\alpha \in [0,1]} \tilde{X}_\alpha \).

A triangular fuzzy number \( \tilde{N} \) is defined by three numbers \( a_1 < a_2 < a_3 \) where the graph of \( \tilde{N}(t) \) is triangle with base on the interval \( [a_1, a_2] \) and vertex at \( a_3 \) where \( \tilde{N}(a_1) = \tilde{N}(a_2) = 0 \), \( \tilde{N}(a_3) = 1 \) and we write \( \tilde{N} = (a_1/a_2/a_3) \).

For \( x \in \mathbb{R}^n \) and \( A, B \subset \mathbb{R}^n \) let
\[
\rho(x,A) = \inf \{ \|x - a\| : a \in A \},
\]
\[
\beta(A,B) = \sup \{ \rho(a,B) : a \in A \},
\]
\[
d_H(A,B) = \max \{ \beta(A,B), \beta(B,A) \}.
\]

The Hausdorff distance \( d_H \) defines a metric on the nonempty and compact subsets of \( \mathbb{R}^n \). For two fuzzy sets \( \tilde{A}, \tilde{B} \) the Hausdorff metric is defined as \( d_H(\tilde{A},\tilde{B}) = \sup_{\alpha \in [0,1]} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha) \).

We can replace functions and initial values in the problem
\[
\dot{x}(t) = f(t,x(t))\quad x(0) = x_0
\]
by set-valued functions which leads to the following differential inclusion (DI),
\[
\dot{x}(t) = F(t,x(t))\quad x(0) = X_0
\]
(1)

Where \( F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\} \) is a set-valued function and \( X_0 \subset \mathbb{R}^n \) is compact and convex. A function \( x : [0,T] \to \mathbb{R}^n \) is a solution of (2) if it is an absolutely continuous and satisfies (2) almost everywhere. Let \( \chi \) denote the set of all solutions of (2), the reachable set \( X(t) \) at time \( t \in [0,T] \) is defined as,
\[
X(t) = \{ x(t) / x \in \chi \}.
\]

The set \( X(t) \) is the set of all possible solutions of (1) at time \( t \). A reasonable generalization of this approach which takes vagueness into account is to replace sets by fuzzy sets, i.e. (2) becomes the fuzzy differential inclusion,
\[
\dot{x}(t) \in \tilde{F}(t,x(t))\quad x(0) \in \tilde{X}_0
\]
(3)

On \( [0,T] \) with a fuzzy function \( \tilde{F} : [0,T] \times \mathbb{R}^n \to E^n \), where fuzzy set \( \tilde{X}_0 \in E^n \) and \( E^n \) is the set of normal, upper semi-continuous, fuzzy convex and compactly supported fuzzy sets on \( \mathbb{R}^n \). Also \( \dot{x}(t) \) is the usual crisp derivative of the crisp differentiable function \( x(t) \) with respect to \( t \). In this paper we introduce a STHW technique for finding reachable set \( \tilde{X}(t) \) that are based on the theoretical consideration of the following theorem.

Theorem 1: Suppose the fuzzy function \( \tilde{F} : [0,T] \times \mathbb{R}^n \to E^n \) to be continuous in \( t \) and also satisfies Lipschitz condition \( d_H(\tilde{F}(t,x),\tilde{F}(t,y)) \leq L|x - y| \) on \( \mathbb{R}^n \) with Lipschitz \( L > 0 \). Consider the set \( \tilde{X} \) of
solutions to (3). The reachable set \( \widetilde{X}(t) \) associated with \( \widetilde{X} \) is a normal, upper semi-continuous and compactly supported fuzzy set for all \( t \in [0, T] \). If \( \widetilde{F} \) is also concave, i.e.,
\[
\alpha \tilde{F}(t, x) + \beta \tilde{F}(t, y) \leq \tilde{F}(t, \alpha x + \beta y),
\]
for all \( \alpha, \beta > 0, \alpha + \beta = 1 \), then \( \widetilde{X}(t) \in E^n \).

Now, consider the initial value problem (3) with \( n=1 \), i.e.
\[
x(t) \in \tilde{F}(t, x(t)) \quad \text{for all } t \in [0, T],
x(0) \in \widetilde{X}_0.
\]

On \( J = [0, T] \) with a fuzzy concave function \( \tilde{F} : J \times R \rightarrow E \), where fuzzy set \( \widetilde{X} \in E \) and the hypotheses of Theorem 1 are satisfied. We call a function \( x_\alpha : J \rightarrow R \) an \( \alpha \)-solution to (4), if it is absolutely continuous and satisfies
\[
\dot{x}_\alpha(t) \in F_\alpha(t, x(t)),
x_\alpha(0) \in \widetilde{X}_0,
\]
almost everywhere on \( J \), where \( F_\alpha(t, x(t)) \) is the \( \alpha \)-cut of the fuzzy set \( \tilde{F}(t, x(t)) \). The set of all \( \alpha \)-solution to (5) is denoted by \( \chi_\alpha \), and the \( \alpha \)-reachable set \( X_\alpha(t) \) is defined as
\[
X_\alpha(t) := \{ x(t) : x \in \chi_\alpha \}.
\]

In this paper, the \( \alpha \)-reachable set \( X_\alpha(t) \) is approximated by STHW technique.

**Properties of Haar wavelet and STHW Technique:**

**Haar Wavelet Series:**

The orthogonal set of Haar wavelets \( h_i(t) \) is a group of square waves with magnitude of \( \pm 1 \) in some intervals and zeros elsewhere. In general,
\[
h_i(t) = h_i(2^j t - k),
\]
where \( n = 2^j + k \), \( j \geq 0, 0 \leq k < 2^j, n, j, k \in Z \).

Any function \( y(t) \), which is square integrable in the interval \( [0, 1) \) can be expanded in a Haar series with an infinite number of terms
\[
y(t) = \sum_{i=0}^{\infty} c_i h_i(t), \quad \text{Where } i = 2^j + k
\]
(6)

Where the Haar coefficients \( j \geq 0, 0 \leq k < 2^j, t \in [0, 1) \)
\[
c_i = \int_0^1 y(t) h_i(t) dt
\]
are determined such that the following integral square error \( \mathcal{E} \) is minimized
\[
\mathcal{E} = \int_0^1 \left[ y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt,
\]
where \( m = 2^j, j \in [0] \cup N \).

Furthermore
\[
\int_0^1 h_i(t) h_l(t) dt = 2^{-j} \delta_{ij} = \begin{cases} 2^{-j}, & i = l = 2^j + k, j \geq 0, 0 \leq k < 2^j \\ 0, & i \neq l \end{cases}
\]

Usually, the series expansion Eq. (6) contains an infinite number of terms for a smooth \( y(t) \). If \( y(t) \) is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (6) will be terminated after \( m \) terms, that is
\[
y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = C_{(m)} h_{(m)}(t), \quad t \in [0, 1)
\]
\[
c_{(m)}(t) = [c_0 c_1 \ldots c_{m-1}]^T, \quad h_{(m)}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T
\]
where “T” indicates transposition, the subscript \( m \) in the parentheses denotes their dimensions, \( C_{(m)}^{T}h_{(m)}(t) \) denotes the truncated sum. Since the differentiation of Haar wavelets results in generalized functions, which in any case should be avoided, the integration of Haar wavelets are preferred.

Integration of Haar Wavelets should be expandable in Haar series

\[
\int_{0}^{t} h_{(m)}(\tau) d\tau \approx E_{(m\times m)}h_{(m)}(t), \quad t \in [0,1]
\]

where the \( m \)-square matrix \( E \) is called the operational matrix of integration which satisfies the following recursive equations

\[
E_{(m\times m)} = \begin{bmatrix} E_{\left(\frac{m}{2}\times \frac{m}{2}\right)} & \left(-\frac{1}{2m}\right)H_{\left(\frac{m}{2}\times \frac{m}{2}\right)}^T \\ \left(\frac{1}{2m}\right)H_{\left(\frac{m}{2}\times \frac{m}{2}\right)} & 0_{\left(\frac{m}{2}\times \frac{m}{2}\right)} \end{bmatrix}
\]

\[
E_{(2\times 2)} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad E_{(1\times 1)} = \frac{1}{2}
\]

The \( H_{m\times m} = \left[h_{n}(x_{0}), h_{n}(x_{1}), h_{n}(x_{2}), \ldots, h_{n}(x_{m-1})\right], \quad \frac{i}{m} \leq x_{i} \leq \frac{i+1}{m} \)

\[
H_{(m\times m)}^{-1} = \left(\frac{1}{m}\right)H_{(m\times m)}^{T} \text{dia}(r),
\]

\[
r = \left[1,1,2,2,4,4,4, \ldots, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \ldots \right]^{T}, \quad m > 2
\]

Proof of equation (7) is found in [7]. Since \( H_{(m\times m)} \) and \( H_{(m\times m)}^{-1} \) contain many zeros. Let us define

\[
h_{(m)}(t)h_{(m)}^{T}(t) \approx M_{(m\times m)}(t), \quad \text{and} \quad M_{(1\times 1)}(t) = h_{0}(t)
\]

and Satisfying \( M_{(m\times m)}(t)c_{(m)} = C_{(m\times m)}h_{(m)}(t) \) and \( C_{(1\times 1)} = c_{0} \).

**Single Term Haar Wavelet series Technique:**

With the STHWS approach, in the first interval, the given function is expanded as STHWS in the normalized interval \( \tau \in [0,1] \), which corresponds to \( \tau \in [0,\frac{1}{m}] \) by defining \( \tau = mt \), \( m \) being any integer.

In STHWS, the matrix becomes \( E = \frac{1}{2} \). Let \( \hat{x}(\tau) \) and \( \hat{X}(\tau) \) be expanded by STHWS in the first interval as \( \hat{x}(\tau) = \psi^{(1)}h_{0}(\tau), \quad \hat{X}(\tau) = x^{(1)}h_{0}(\tau) \) and in the \( n^{\text{th}} \) interval as \( \hat{x}(\tau) = \psi^{(n)}h_{0}(\tau), \quad \hat{X}(\tau) = x^{(n)}h_{0}(\tau) \)
Integrating (2.12) with $E = \frac{1}{2}$, we get $x^{(i)} = \frac{1}{2} \nu^{(i)} + x(0)$. Where $x(0)$ is the initial condition.

According to [7], we have $\nu^{(i)} = \int x(\tau) d\tau = x(1) - x(0)$.

In general, for any interval $n$, $n = 1, 2, \ldots$.

We obtain, $x^{(n)} = \frac{1}{2} v^{(n)} + x(n - 1)$ (8)

$\dot{x}(n) = v^{(n)} + x(n - 1)$ (9)

Equation (8) and (9) give the discrete time values of $x^{(n)}$ and $x(n)$ for the $n$th interval. These values from the basis for the estimating block pulse values and discrete values in the subsequent normalized time intervals.

Numerical example:

In this section, we apply STHW to solve one dimensional fuzzy differential inclusion. The main objective here is to solve this example using the STHW given in Section 2 and compare our results with the presented results in E.Babolian, S.Abbasbandy and M.Alavi (2009). Consider the fuzzy differential inclusions on $R^+$,

$$
\begin{cases}
\dot{x}(t) \in -x(t) + \bar{e} \cos t \\
x(0) \in \bar{X}_0,
\end{cases}
$$

where $\bar{e}$ and $\bar{X}_0$ are symmetric triangular fuzzy numbers with level sets $[\bar{e}]_\alpha = [0.05(\alpha - 1), 0.05(1 - \alpha)]$ and $[\bar{X}_0]_\alpha = [0.05(\alpha - 1), 0.05(1 - \alpha)]$. The $\alpha$-solution set is given for $t \geq 0$ by $x_\alpha(t) = \frac{1}{2}(\sin t + \cos t) \bar{e} + \left( [\bar{X}_0]_\alpha - \frac{1}{2} [\bar{e}]_\alpha \right) e^\alpha$.

Now, we obtain the approximation using Euler and STHW of 0-reachable set and calculated error in Figure 1 and $\bar{X}(5)X$ in Figure 2 with $\Delta t = 0.01$.

![Fig. 1: Error estimation of 0-reachable set $[\bar{X}_0]_\alpha$](image-url)
Fig. 2: Error estimation of $\widetilde{X}(5)$

From the graphical representation is given for the one dimensional fuzzy differential inclusion shows that Euler solutions (E.Babolian, S.Abbasbandy and M.Alavi (2009)) have little error in the all the stages but STHW method approximate solutions match well in all stages.

Conclusions:

In this paper, the single-term Haar wavelet series (STHW) technique has been successfully employed to obtain the approximate analytical solutions of the one dimensional fuzzy differential equation based on fuzzy differential inclusion.

Compare to Euler method, STHW technique gives accurate results from the figure 1 and 2. By STHW method we can approximate fuzzy $\alpha$ - reachable set, since $x_{\alpha}(t)$ is an interval. Numerical example shows the efficiency of implemented STHW technique.

References


