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Spline/Spectral Methods for Neutral Volterra Integro-Differential Equations with Delay

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ABSTRACT

This paper presents a new technique for numerical solution of neutral Volterra integro-differential equations with delay that have many applications in biological and physical sciences. The technique is based on a combination of quintic spline collocation and El-Gendi method. Numerical results illustrating the efficiency of the presented method when faced with some difficult examples are presented.

Key words: Neutral Volterra integro-differential equations; Quintic spline; Collocation methods; El-Gendi method.

Introduction

We consider neutral Volterra integro-differential equations with delay \( \tau \)

\[
y'(t) = f(t, y(t)) + \int_{t-\tau}^{t} K(t, v, y(v), y'(v)) \, dv, \quad t \in [t_0, T],
\]

(1.a)

with the initial condition

\[
y(t) = \phi(t), \quad t \in [-\tau, t_0],
\]

(1.b)

where \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) and \( K : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), \( \tau \) is a positive number, and \( \phi \) is a given \( C^1 \)-function. Note that, when \( \tau = 0 \), Eq. (1) reduces to a standard initial value problem. Numerical simulation of such equations is becoming more and more important as they are used more and more for describing complex systems in physical and biological phenomena; we refer the reader to many examples to the monograph (Brunner, H., 1993). A wide variety of numerical methods for neutral Volterra integro-differential equations with delay have been presented in the past (see, e.g. (Brunner, H. and P.J. van der Houwen, 1986; Brunner, H., 1993; Enright, W.H. and Min Hu, 1997; Vermiglio, R., 1988). Spline collocation methods for solving delay and neutral delay differential equations were studied in (El-Hawary, H.M. and S.M. Mahmoud, 2003; El-Hawary, H.M. and K.A. El-Shami, 2009; El-Hawary, H.M. and K.A. El-Shami, 2012). More detailed analysis for both the convergence and absolute stability was also given.

In this paper, we present a mixed spline/spectral method to solve the neutral Volterra integro-differential equations with delay (1). This paper is organized as follows. In Section 2, we gave the basic idea of El-Gendi Method. The description of spline/spectral methods for the numerical solution of Eq. (1) are presented in Section 3. In Section 4, three numerical results for both linear and non-linear examples are solved to illustrate the efficiency of the presented method. The last section is conclusion.

2 El-Gendi Method:

The use of integration operators for the treatment of differential equations by orthogonal polynomials dates back to Clenshaw (1957), Integration form of differential equations was applied by Clenshaw and Curtis (1960) in spectral space and by El-Gendi (1969) in point space. El-Gendi (1969) has extensively shown how Chebyshev expansions can be used to solve linear integral equations, integro-differential equations and ordinary differential equations. Also, Delves and Mohamed (1985) have shown that El-Gendi method represents a modification of the Nystrom scheme when applied to solving Fredholm integral equations of the second kind (Elsayed M.E. Elbarbary, 2007).

Clenshaw and Curtis (1960) gave a procedure for the numerical integration of a nonsingular function \( f(x) \) which is defined on a finite range \(-1 \leq x \leq 1\) by expanding the function in a series of Chebyshev polynomials.
as follows:

\[(P_N f)(x) = \sum_{k=0}^{N} a_k T_k(x),\] (2)

where

\[a_k = \frac{2}{N} \sum_{j=0}^{N} f(x_j) T_k(x),\] (3)

and integrating this series term by term. The summation symbol with double primes denotes a sum with both the first and last terms halved. In (El-Gendi, S.E., 1969), the author proposes an integration matrix \(B\) to approximate the indefinite integral as follows:

\[\int_0^\tau (P_N f)(t) dt = \sum_{j=0}^{N} a_j \int_0^\tau T_j(t) dt = \sum_{j=0}^{N+1} \hat{c}_j T_j(x),\] (4)

where

\[\hat{c}_0 = \sum_{j=0, j \neq 1}^{N} \frac{(-1)^{j+1} a_j}{j^2 - 1} - \frac{1}{4} a_1,\]

\[\hat{c}_k = \frac{a_{k-1} - a_{k+1}}{2k}, \quad k = 1(1)N - 2,\]

\[\hat{c}_{N-1} = \frac{a_{N-2} - 0.5a_N}{2(N-1)},\]

\[\hat{c}_N = \frac{a_{N-1}}{2N},\]

\[\hat{c}_{N+1} = \frac{a_N}{4(N+1)}\]

after certain arrangements we arrive to

\[\int_0^\tau (P_N f)(t) dt = B[f],\]

where \(B = [b_{ij}]\) is a square matrix of order \((N + 1)\), \(b_{ij}\) are the elements of the matrix \(B\), and the elements of the column matrix \([f]\) are given by \(f_k = f(x_k)\), where \(x_k\) are the Gauss-Lobatto points

\[x_k = -\cos\left(\frac{k\pi}{N}\right), \quad k = 0(1)N\] (5)

3 Description of the Methods:

Consider the initial value problem for the neutral Volterra integro-differential equations with delay (1). For a given positive integer \(n\), the interval \([t_0, T]\) is partitioned into \(n\) equal subintervals \(I_i = [t_{i-1}, t_i], i = 1(1)n\) with \(t_i = t_{i-1} + h, n = (T - t_0)/h\), \(h\) is the stepsize. The basic idea is to generate a quintic spline collocation methods \(S \in C^4[t_0, T]\) at the Chebyshev points

\[c_i = \frac{t_i + t_{i-1}}{2} + \frac{t_i - t_{i-1}}{2} (-\cos \frac{\ell \pi}{4}), \quad \ell = 0(1)4, i = 1(1)n.\] (6)
Let $S^{(1)}_{n,5} = \{S(t) : S \in C^{1}[t_0,t_f], S \in \Pi_5, \text{ for } t \in I_i, i = 1(1)n\}$, where $\Pi_5$ denotes the collection of all polynomials of degree $\leq 5$. Using the notations

$$
S'_{i-1} = S'(t_{i-1}), \quad S'_{i-1+c_1} = S'(t_{i-1+c_1}), \quad S'_{i-1+c_2} = S'(t_{i-1+c_2}),
$$

$$
S'_{i-1+c_3} = S'(t_{i-1+c_3}), \quad S'_{i} = S'(t_i), i = 1(1)n,
$$
a quintic spline functions $S \in S^{(1)}_{n,5}$ can be represented on each $I_i$ by

$$
S_i(t) = S_{i-1} + hA(\xi)S'_{i-1} + hB(\xi)S'_{i-1+c_1} + hC(\xi)S'_{i-1+c_2} + hD(\xi)S'_{i-1+c_3} + hE(\xi)S'_{i},
$$

where $A(\xi),...,E(\xi)$ are given in the Appendix, $t = t_{i-1} + \xi h, \xi \in [0,1]$. Since $S \in S^{(1)}_{n,5}$, then the approximate spline solution $S(t)$ to the exact solution $y(t)$ of Eq. (1) will be constructing as follows: for $i = 1(1)n$

$$
S_i = M_0S_{i-1} + hM_1S'_{i-1} + hM_2S'_{i},
$$

where $S_i = (S_{i-1+c_1},S_{i-1+c_2},S_{i-1+c_3},S_{i})^T, \quad S'_{i} = (S'_{i-1+c_1},S'_{i-1+c_2},S'_{i-1+c_3},S')^T,$

$$
S'_{i-1+c_1} = f(t_{i-1+c_1},S(t_{i-1+c_1}))+\int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S(v),S'(v))dv,
$$
c_i be given in Eq. (7), $M_0 = (1,1,1,1)^T$, $M_1$ and $M_2$ are also given in the Appendix of this paper.

To find an approximation to the integral

$$
\int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S(v),S'(v))dv,
$$
we subdivide its integration interval as follows: if $(t_{i-1+c_1} - \tau) \leq t_0$

$$
\int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S(v),S'(v))dv = \int_{t_{i-1+c_1}}^{t_0}K(t,v,S(v),S'(v))dv
$$

$$
+ \int_{t_0}^{t_{i-1+c_1}}K(t,v,S(v),S'(v))dv
$$

$$
= \int_{t_{i-1+c_1}}^{t_0}K(t,v,\phi(v),\phi'(v))dv
$$

$$
+ \int_{t_0}^{t_{i-1+c_1}}K(t,v,S_{m+1}(v),S_{m+1}'(v))dv
$$

$$
+ \int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S_{i}(v),S_{i}'(v))dv,
$$

when $t_0 < (t_{i-1+c_1} - \tau) \in [t_{k-1},t_k], k = 1(1)i$

$$
\int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S(v),S'(v))dv = \int_{t_{i-1+c_1}}^{t_0}K(t,v,S_k(v),S_k'(v))dv
$$

$$
+ \int_{t_0}^{t_{i-1+c_1}}K(t,v,S_{m+1}(v),S_{m+1}'(v))dv
$$

$$
+ \int_{t_{i-1+c_1}}^{t_{i-1+c_2}}K(t,v,S_{i}(v),S_{i}'(v))dv,
$$
Each of the above integrals in Eq. (9) is approximated by applying El-Gendi method (see above Section 2). Since El-Gendi method is defined for the finite range \(-1 \leq t \leq 1\), then all the integral subintervals in Eq. (9) must be converted to \([-1,1]\), as example:

\[
\int_{i-1}^{i} K(t,v,S_i(v),S'_i(v))dv = \int_{i-1}^{i} K(t,w,S_i(w),S'_i(w))dw,
\]

where

\[
w = \frac{t_i - t_{i-1} + t_i + t_{i-1}}{2}, \quad dw = \frac{t_i - t_{i-1}}{2} dv.
\]

From Eq. (9) and (10), system (8) can be solved for \(S_{i-1+1}, S_{i-1+2}, S_{i-1+3}, S_i\).

4 Numerical Examples:

To illustrate our discussion, three test examples will be considered. We can compute their actual error and compare the performance of the above mentioned method. The computer application program MATLAB 7.1 was used to execute the algorithms that were used to solve the given examples. We choose \(N = 8\) in Eq. (5) for each of the following examples.

**Example 4.1** (Enright, W.H. and Min Hu, 1997) Consider the following linear equation:

\[
y'(t) = -te^{t^2}(-90 + 9\alpha - 10t + 10\tau + \alpha t - \alpha \tau)y(t)
\]

\[
+ (-0.1 - 90t + 9\alpha t - 10t^2 + \alpha^2)y(t)
\]

\[
+ \int_{t_{i-1}}^{t} \tau(1-v)(y(v) + \alpha y'(v))dv, \quad t \in [0,4],
\]

\[
y(t) = e^{-t^2}, \quad t \leq 0,
\]

with the exact solution \(y(x) = e^{-x^2}\), where \(\alpha\) and \(\tau > 0\) are constants.

In Table 1, we give the absolute errors between the exact solution and the numerical results by the present method for \(\tau = 1\) and different values of \(h, t, \alpha\).

**Example 4.2** (Enright, W.H. and Min Hu, 1997) Consider the following non-linear equation:

\[
y'(t) = \sin(t - \frac{1}{2}) + \int_{\frac{1}{2}}^{t} \cos(v)([y'(v)]^2 + [y(v)]^2)dv, \quad t \in [0,10],
\]

\[
y(t) = -\cos(t), \quad t \leq 0,
\]

with the exact solution \(y(t) = -\cos(t)\).

Figure 1 shows both the approximate solution by the present method and the exact solution for \(h = 0.2\). In (Enright, W.H. and Min Hu, 1997), the continuous Runge-Kutta methods were used. It showed that the maximum absolute error is \(1.88 \times 10^{-8}\) for tolerance \(10^{-8}\). In Table 2, we give the absolute errors between the exact solution and the numerical results by the present method for \(h = 0.1\). We used Newton method with tolerance \(10^{-8}\) for solving nonlinear systems.

**Example 4.3** (Enright, W.H. and Min Hu, 1997) Consider the following non-linear equation:

\[
y'(t) = 2t + \frac{1}{2}te^{t^2}(e^{(1-2t)} - 1) + \int_{t_{i-1}}^{t} tv(e^{y(t)} + \frac{v}{10}y'(t) - \frac{1}{5} y(v))dv, \quad t \in [0,2],
\]

\[
y(t) = t^2, \quad t \leq 0,
\]

with the exact solution \(y(t) = t^2\).

In (Enright, W.H. and Min Hu, 1997), the continuous Runge-Kutta methods were used. It showed that the maximum absolute error is \(9.63 \times 10^{-6}\) for tolerance \(10^{-7}\). In Table 3, we give the absolute errors between the exact solution and the numerical results by the present method for \(h = 0.2\). We used Newton method with tolerance \(10^{-4}\) for solving nonlinear systems.
Table 1: Absolute errors for the solution of Example 4.1 with $\tau = 1$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$I_i$</th>
<th>$h = 0.5$</th>
<th>$h = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>8.8817841E-15</td>
<td>5.3290705E-15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>4.1078251E-14</td>
<td>1.3766765E-14</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>6.7208456E-14</td>
<td>3.3972624E-14</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9.2069459E-14</td>
<td>8.9706020E-14</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>8.582039E-14</td>
<td>5.9952043E-15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.1912693E-13</td>
<td>1.0436096E-14</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.0946799E-13</td>
<td>1.0214051E-14</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9.9587005E-14</td>
<td>8.8817841E-15</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>4.3942627E-13</td>
<td>1.4432899E-15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.9089178E-13</td>
<td>2.8865998E-15</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>6.737903E-13</td>
<td>2.2304460E-15</td>
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<tr>
<td></td>
<td>4</td>
<td>5.8046664E-13</td>
<td>1.6653345E-15</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>6.1024518E-12</td>
<td>9.3702823E-14</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.0662470E-11</td>
<td>1.6453505E-13</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>9.8622221E-12</td>
<td>1.5298738E-13</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>8.9909191E-12</td>
<td>1.4011014E-13</td>
</tr>
</tbody>
</table>

Fig. 1: The approximate solution and exact solution of Example 4.2 for $h = 0.2$.

Table 2: Absolute errors for the solution of Example 4.2 with $h = 0.1$.

<table>
<thead>
<tr>
<th>$I_i$</th>
<th>Absolute errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.7926554E-13</td>
</tr>
<tr>
<td>2</td>
<td>1.1517453E-12</td>
</tr>
<tr>
<td>3</td>
<td>2.1715962E-13</td>
</tr>
<tr>
<td>4</td>
<td>7.9791728E-13</td>
</tr>
<tr>
<td>5</td>
<td>1.2946865E-12</td>
</tr>
<tr>
<td>6</td>
<td>4.6673775E-13</td>
</tr>
<tr>
<td>7</td>
<td>6.5769811E-13</td>
</tr>
<tr>
<td>8</td>
<td>1.4146461E-12</td>
</tr>
<tr>
<td>9</td>
<td>7.8692607E-13</td>
</tr>
<tr>
<td>10</td>
<td>4.5929926E-13</td>
</tr>
</tbody>
</table>
Table 3: Absolute errors for the solution of Example 4.3 with h=0.2.

<table>
<thead>
<tr>
<th>$I_j$</th>
<th>Absolute errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>2.5656411E-09</td>
</tr>
<tr>
<td>0.4</td>
<td>3.3101901E-09</td>
</tr>
<tr>
<td>0.6</td>
<td>3.3874760E-09</td>
</tr>
<tr>
<td>0.8</td>
<td>3.3821514E-09</td>
</tr>
<tr>
<td>1.0</td>
<td>3.3322518E-09</td>
</tr>
<tr>
<td>1.2</td>
<td>3.252871E-09</td>
</tr>
<tr>
<td>1.4</td>
<td>3.106228E-09</td>
</tr>
<tr>
<td>1.6</td>
<td>3.081375E-09</td>
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<tr>
<td>1.8</td>
<td>3.0382051E-09</td>
</tr>
<tr>
<td>2.0</td>
<td>2.9863251E-09</td>
</tr>
</tbody>
</table>

5 Conclusion:

This paper presented a new technique for the solution of the neutral Volterra Integro-differential equations with delay. The technique is based on a combination of $C^1$-spline collocation and El-Gendi method. The proposed method is applied to solve linear and non-linear examples. Numerical results have been used to demonstrate the efficiency and accuracy of the proposed method.

Acknowledgment

The authors are indebted to Professor S.E. El-Gendi for various valuable suggestions and constructive criticism.

References


Appendices

A. Appendix In this Appendix, we give the $A(\xi), \dot{E}(\xi)$.
\[ A(\xi) = \xi - \frac{11}{2} \xi^2 + \frac{34}{3} \xi^3 - 10 \xi^4 + \frac{16}{5} \xi^5, \]
\[ B(\xi) = 6.82842712474616 \xi^2 - 18.990187582825666 \xi^3 + 18.82842712474616 \xi^4 - \frac{32}{5} \xi^5, \]
\[ C(\xi) = -2 \xi^2 + 12 \xi^3 - 16 \xi^4 + 6.4 \xi^5, \]
\[ D(\xi) = 1.17157287525384 \xi^2 - 7.6764790834101 \xi^3 + 13.17157287525384 \xi^4 - 6.4 \xi^5, \]
\[ E(\xi) = -0.5 \xi^2 + \frac{10}{3} \xi^3 - 6 \xi^4 + \frac{16}{5} \xi^5. \]

**B. Appendix** In this Appendix, we give the \( \mathbf{M}_1 \) and the matrix \( \mathbf{M}_2 \).

\[
\mathbf{M}_1 = \begin{bmatrix}
0.09503171601907 & -0.01213203435596 & 0.00664336837074 & 0.00279822031356 \\
0.31011002862997 & 2.0 & -0.04344336196330 & 0.0166666666667 \\
0.26002329829592 & 0.41213203435597 & 0.1716349564760 & -0.02636844635311 \\
4/15 & 2/5 & 4/15 & 1/30
\end{bmatrix}
\]

\[
\mathbf{M}_2 = \begin{bmatrix}
0.31011002862997 & 2.0 & -0.04344336196330 & 0.0166666666667 \\
0.26002329829592 & 0.41213203435597 & 0.1716349564760 & -0.02636844635311 \\
4/15 & 2/5 & 4/15 & 1/30
\end{bmatrix}
\]