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Analytical approach for the determination of complex potential and pressure in the production and reinjection wells of a geothermal reservoir

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ABSTRACT

This paper intends to investigate the velocity of the complex potential and the pressure of fluid of the geothermal reservoir of production and reinjection wells. Using Bernoulli’s numbers, we determine the velocity of the complex potential of the geothermal reservoir as a sum of convergent series:

\[ \tilde{q} = \frac{Q}{2\pi l} \left\{ \coth \left( \frac{z-l}{t} \right) - \coth \left( \frac{z+l}{t} \right) \right\} \text{ for } |z-l| < t, |z+l| < t \]

Then the pressure of the geothermal fluid in production well and reinjection well is computed.

Key words: complex potential, geothermal field, Bernoulli’s numbers, filtration velocity, sum of convergent series.

Introduction

The groundwater mechanics remains one of the most important problem for the fluid mechanics in the porous medium. This applied fluid mechanics is the main part of the petroleum engineering knowledge.

In the recent years, water resources management includes groundwater management. The flow of low temperature groundwater through interconnections of aquifers can be considered as laminar flow in fluids mechanics. In geothermal reservoirs apart from the transient phase, flows depend on time and at given time can be considered as permanent flows and then can be characterized by the velocity (Bobok, E., 1987). The potential of velocity field exists if the fluid flow is irrotational (Smirnov, V., 1979), i.e.

\[ \text{rot} \tilde{q} = 0 \]  \hspace{1cm} (1)

Consider the well-known identity in the vector analysis:

\[ \text{rot} (\text{grad} \phi) = 0 \]  \hspace{1cm} (2)

From the relations (1), (2) it follows

\[ \tilde{q} = \text{grad} \phi \]  \hspace{1cm} (3)

This hypothesis is satisfied with the Helmhotz-Rayleigh conditions; hence we are in the case of laminar incompressible fluid flow in the homogeneous isotropic and porous reservoir. From the equations (1), (2) and (3) we obtain a similar expression as in (Smirnov, V., 1979).

\[ \text{rot} (\text{rot} \tilde{q}) = \text{grad} \phi \]  \hspace{1cm} (4)

Using the conservation balance of the circulation \( \Gamma \) (Bobok, E., 1987) we get

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\[
\frac{d\Gamma}{dt} = \frac{d}{dt} \int_V q dS = 0.
\] (5)

Then
\[
\tilde{q} = \text{grad}\phi = -\frac{q_k}{v} \text{grad}\left(h + \frac{p}{\rho g}\right)
\] (6)

and the potential velocity becomes
\[
\phi = -\frac{q_k}{v} \left(h + \frac{p}{\rho g}\right)
\] (7)

For this analysis we suppose that the reservoir is homogeneous, isotropic with constant thickness supplying a geothermal system containing production and re-injection wells (Boldizsar, T., 1968), (Bobok, E., 1987). This analytical approach proposes a mathematical model of the complex potential of the velocity field, the current and the pressure in the wells. The rest of this paper is organized as follows: In the next section we adopt the assumptions and the complex potential principles are developed. We compute the filtration velocity. In the section 3 we computed the static pressure of the geothermal reservoir at the bottom of the production and re-injection wells and by the same way, the pressure at the head of production well. Finally we concluded.

2 Determination of the complex potential and the filtration velocity:

Consider an infinite reservoir homogeneous isotropic with constant thickness supplying \( n \) production and \( m \) re-injection wells (Bobok, E., 1987). In the Cartesian frame of reference, \( z_v \) stands for the coordinate of the production well and \( z_\mu \) for the re-injection well. Then the complex potential of the wells system is expressed as:
\[
w = -\sum_{i=1}^{n} \frac{Q_i}{2\pi} \ln(z - z_v) + \sum_{\mu=1}^{m} \frac{Q_\mu}{2\pi} \ln(z - z_\mu)
\] (8)

where \( Q \) is the intensity.

At each point of coordinate \( z \), the filtration velocity \( \tilde{q} \) is defined as \( \tilde{q} = \frac{dw}{dz} \), where \( \tilde{q} \) is the conjugate of \( q \).

And
\[
q = -\frac{Q}{2\pi} \sum_{i=1}^{n} \frac{1}{z - z_v} + \frac{Q}{2\pi} \sum_{\mu=1}^{m} \frac{1}{z - z_\mu}
\] (9)

In order to determine the hydrodynamic conditions of the reservoir with the network of wells, we considered these wells as nodes. At each node we can determine the velocity, the pressure (piezometric level) and the direction of streamlines (Schlichting, H., 1968). The experiences of the reservoir mechanics show that the disposition of the geothermal wells is similar to that of wells in petroleum engineering. Hence, the usual disposition of the geothermal wells is such that the rate of flow of the production wells is identical, the capacity of re-injection wells is similar and the distance between two arbitrary wells is constant and equal to \( t \) (Boldizsar, T., 1968). The production of fluid in these wells can reach an optimal value. Consider the geothermal system of \( N \) production wells and \( N \) re-injection wells at constant distance \( t \) each other as nodes (Bobok, E., 1987), lying on the straight lines whose equations are \( x = l \) and \( x = -l \) respectively. Then the complex potential can be defined as
\[
w = \frac{Q}{2\pi} \sum_{i=1}^{N} \frac{1}{z - z_i} \left[ \ln(z - l + ivt) - \ln(z + l + ivt) \right].
\] (10)
The velocity of filtration $q$ is expressed by 
\[
q = \frac{Q}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{z-l+ivt} - \frac{1}{z+l+ivt} \right]
\]  
and the distribution of the field velocity is repetitive with period $t$.

Using the theory of complex numbers (Duncan, J., 1974) and supposing that the model contains a countable number of wells (Asszony, G., 1974) the complex potential can be defined as
\[
w = \frac{Q}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \ln(z-l+ivt) - \ln(z+l+ivt) \right]
\]  
(12)

And
\[
q = \frac{Q}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{z-l+ivt} - \frac{1}{z+l+ivt} \right]
\]  
(13)

Out of the immediate environment of the well, the flow velocity distribution is similar to parallel flow model (Toth, J., 1979). Let us define the sum of the series of the relation (13). Thus we can write
\[
q = \frac{Q}{2\pi} \left[ \frac{2l}{z^2-l^2} + \frac{2(z-l)}{z-l} \sum_{n=1}^{\infty} \frac{1}{v^2t^2} - 2(z+l) \sum_{n=1}^{\infty} \frac{1}{v^2t^2} \right]
\]  
(14)

Taking into account
\[
\frac{u^2}{u^2 + v^2 t^2} = -\sum_{n=1}^{\infty} \left( \frac{-u^2}{v^2 t^2} \right)^n |z| < vt
\]  
(15)

the last relation can be reduced to
\[
q = \frac{Q}{2\pi} \left[ \frac{2l}{z^2-l^2} - \frac{2}{z-l} \sum_{n=1}^{\infty} \frac{1}{v^2t^2} \left( \frac{-z-l}{v^2t^2} \right)^n + \frac{2}{z+l} \sum_{n=1}^{\infty} \frac{1}{v^2t^2} \left( \frac{-z+l}{v^2t^2} \right)^n \right]
\]  
(16)

\[|z-l| < vt, |z+l| < vt\]  
(17)
equivalent to
\[
q = \frac{Q}{2\pi} \left[ \frac{2l}{z^2-l^2} - \frac{2}{z-l} \sum_{n=1}^{\infty} (-1)^m s_2m \left( \frac{z-l}{t} \right)^{2m} + \frac{2}{z+l} \sum_{n=1}^{\infty} (-1)^m s_2m \left( \frac{z+l}{t} \right)^{2m} \right]
\]  
(18)

\[s_p = \sum_{n=1}^{\infty} \frac{1}{\nu^n} \]  
(19)

Taking into account the Bernouilli's numbers $B_n = \frac{2(2m)!}{(2\pi)^{2m}} s_{2m}$ we get
\[s_{2m} = \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}, m \geq 1\] Consequently, we obtain
Finally we obtain the filtration velocity in the form

\[
q = \frac{Q}{2\pi} \left\{ \frac{2l}{z^2 - l^2} - \frac{1}{z} \sum_{m=1}^{\infty} (-1)^m \frac{B_{2m}}{(2m)!} \left( \frac{2\pi(z-l)}{t} \right)^{2m} \right\}
\]

\[
= \frac{Q}{2\pi} \left\{ \frac{2l}{z^2 - l^2} + \frac{1}{z} \left[ \frac{2\pi(z-l)}{t} \frac{1}{e^{\frac{2\pi(z-l)}{t}} - 1} + \frac{1}{2} \frac{2\pi(z-l)}{t} \right] \right\}
\]

\[
- \frac{1}{z+l} \left[ \frac{2\pi(z+l)}{t} \frac{1}{e^{\frac{2\pi(z+l)}{t}} - 1} + \frac{1}{2} \frac{2\pi(z+l)}{t} \right] \right\}
\]

\[
= \frac{Q}{t} \left[ \frac{1}{e^{\frac{2\pi(z+l)}{t}} - 1} - \frac{1}{e^{\frac{2\pi(z-l)}{t}} - 1} \right] \mid z-l \mid \leq t, \mid z+l \mid \leq t \]  

(20)

Finally we obtain the filtration velocity in the form

\[
\bar{q} = \frac{Q}{2r} \left[ \coth \left( \frac{\pi}{t} (z-l) \right) - \coth \left( \frac{\pi}{t} (z+l) \right) \right] \mid z-l \mid \leq t, \mid z+l \mid \leq t \]

(22)

Remark. The relation (22) is useful for using.

Determination of the pressure in the geothermal wells:

\( a. \) The static pressure in the geothermal production and re-injection wells:

Consider two small closed discs \( K_1, K_2 \) defined as

\[
K_1 = \left\{ z \in \mathbb{R} : 0 < \mid z - l \mid \leq t \varepsilon_1 \right\}, \quad K_2 = \left\{ z \in \mathbb{R} : 0 < \mid z + l \mid \leq t \varepsilon_2 \right\},
\]

(23)

where \( \varepsilon_1, \varepsilon_2 \) are two positive small scalars such that \( \varepsilon_1 \leq \varepsilon_2, \varepsilon_2 \leq t > 0 \). We assume that the production and re-injection wells lie on the centers of \( K_1 \) and \( K_2 \) respectively and the initial pressures \( p_{01} \) and \( p_{02} \) are constant and uniformly distributed in \( K_1 \) and \( K_2 \) respectively. Then we set

\[
p(x,y) \big|_{\partial K_1} = p_{01}, \quad p(x,y) \big|_{\partial K_2} = p_{02}
\]

(24)

where \( p(x,y) \) is the static pressure of the flow in motion at each point \( (x,y) \) of the fluid domain.

Thus, the static pressure is defined as

\[
p(x,y) = p_{01} + \frac{Q^2}{8rt^2} \left[ \coth^2 \left( \pi \varepsilon_1 \right) - \frac{1 + \tanh^2 \left[ \frac{\pi}{t} (x-l) \right]}{\tanh^2 \left[ \frac{\pi}{t} (x-l) \right] + \tan^2 \left[ \frac{\pi}{t} y \right]} \right] \]

(25)

\[
p(x,y) = p_{02} + \frac{Q^2}{8rt^2} \left[ \coth^2 \left( \pi \varepsilon_2 \right) - \frac{1 + \tanh^2 \left[ \frac{\pi}{t} (x+l) \right]}{\tanh^2 \left[ \frac{\pi}{t} (x+l) \right] + \tan^2 \left[ \frac{\pi}{t} y \right]} \right] \]

(26)
in the geothermal production and re-injection wells respectively.

**Remark.** The pressure \( p \) is obtained from the relation
\[
\rho(x, y) = C - \frac{1}{2} \|q\|^2, \quad C = \text{const}\]  
(Smirnov, V., 1979).

**b. The pressure ahead of the geothermal production wells:**

In order to determine the pressure ahead of the geothermal production wells, consider the production well of geothermal field whose symmetry axis is shifted to the vertical axis \( z \) directed down with the coordinate origin at the surface. The pressure drop due to the friction of an incompressible fluid moving in a vertical tubing may be computed with the so-called Weisbash formula (Codo, F.P., 1989).

\[
\Delta p = \lambda \rho \frac{v^2 H}{2d}
\]  
(27)

where \( \Delta p \) is the pressure drop, \( \lambda \) a friction coefficient, \( \rho \) a fluid density, \( v \) a fluid velocity in the well, \( H \) the depth and \( d \) an intern diameter of the tubing.

According to Bernouilli (Codo, F.P., 1989), we obtain the balance of kinetic energy for the production system as.

\[
\frac{v_1^2}{2g} + \frac{p_1}{g \rho} - Z_1 = \frac{v_2^2}{2g} + \frac{p_2}{g \rho} + \Delta p - Z_1
\]  
(28)

where \( p_1 \) is the pressure ahead of the well, \( p_2 \) the bottom pressure of the well, \( v_1 \) the fluid velocity ahead of the well, \( v_2 \) the fluid velocity at the bottom of the well and \( Z_1, Z_2 \) are the depths ahead and at the bottom of the well respectively.

For the well known value of the friction coefficient \( \lambda = 0.03 \) the depth \( H = 2000 m \) and the diameter \( d = 0.16 m \), the coefficient \( \lambda \frac{H}{d} \) of the kinetic energy from the Weisbash formula can be computed as:

\[
\lambda \frac{H}{d} = 0.03 \frac{2000}{0.16} = 375
\]

it means that the pressure drop is 375 bigger than the kinetic energy (Codo, F.P., 1989). Thus we can suppose that

\[
\frac{v_1^2}{2} = \frac{v_2^2}{2} \quad \text{or} \quad \frac{v_1^2 - v_2^2}{2} = 0
\]  
(29)

Then the balance (28) becomes

\[
\frac{p_2}{g \rho} = \frac{p_1}{g \rho} + \Delta p + H \quad \text{or} \quad p_1 = p_2 - g \rho H - \lambda \rho H \frac{v_2^2}{2d}
\]  
(30)

where \( H = Z_2 - Z_1 \).

So the pressure at the bottom of the well is computed (Codo, F.P., 1989)

\[
p_2 = p_w - \frac{\mu \delta}{2 \pi kh} \ln \left( \frac{r_i}{r_w} \right)
\]  
(31)

where \( r_i \) is the radius of surface of recharge, \( h \) the thickness of the productive layer, \( r_w \) the radius of the casing, \( k \) the permeability of the reservoir, \( \delta = \pi \nu \frac{d^2}{4} \) the well flow and \( p_w = p(x, y) \) is the static pressure of the reservoir defined by
\[ p_{st} = p_{0t} + \frac{Q^2}{8 \lambda} \left[ \coth^2 \left( \pi \frac{x}{l} \right) - \frac{1 + \tanh^2 \left( \pi \frac{y}{l} \right)}{\tanh^2 \left( \pi \frac{x}{l} \right) + \tan^2 \left( \pi \frac{y}{l} \right)} \right] \]  

which allows to write (30) in the form

\[ p_i = p_{st} - \frac{\mu \delta}{2 \pi \kappa \hbar} \ln \left( \frac{r_i}{r_w} \right) - gH \rho - \lambda \rho \frac{v^2 H}{2d} \]  

In the case of turbulent flow through the wells and for a chosen relative rugosity of the tubings, we use the following formula (Codo, F.P., 1989) \( \lambda = \frac{0.086}{\operatorname{Re}^{0.2}} \) where \( \operatorname{Re} \) is the Reynolds number. Replacing the Reynolds number by \( \operatorname{Re} = \frac{vd}{\nu} \) and \( v = \frac{4\delta}{\pi d^2} \) the quantity \( \lambda \) can be expressed as

\[ \lambda = 0.086 \left( \frac{\pi}{4} \right)^{0.2} \frac{d^{0.2}}{\delta^{0.2}} \nu^{0.2} \]  

where \( \mu \) is the kinematic viscosity of the fluid and then the Weisbash formula for the pressure drop becomes

\[ \Delta p = 0.008194 \rho \frac{d^{0.2}}{\delta^{0.2}} H \frac{16\delta^2}{2d \pi^5 d^4} = 0.06642 \rho \frac{H}{d^{0.8}} \nu^{0.2} \delta^{1.8} \]  

Setting \( C = 0.06642 \rho \frac{H}{d^{0.8}} \nu^{0.2} \), \( D = \frac{\mu}{2 \pi \kappa \hbar} \ln \left( \frac{nt}{r_i} \right) \) we obtain the pressure ahead of geothermal production well in the form

\[ p_i = p_{st} - g \rho H - D\delta - C\delta^{1.8} \]  

Conclusion:

In this paper we have investigated the hydrodynamic behavior of geothermal production and re-injection wells of the geothermal reservoir. Taking into account the Bernouilli's numbers we computed the velocity of the complex potential as the sum (hyperbolic cotangente) of convergent series and the pressure ahead of the geothermal production well has been determinated. The results of this study are the contribution for the analysis of the geothermal system constituted by the reservoir and the production and re-injection wells.

References