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ORIGINAL ARTICLES

New Third-order Iterative Method for Solving Nonlinear Equations.

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ABSTRACT

In this paper, we present a new third-order iterative method for solving nonlinear equations. The new method is based on Newton-Raphson method and Taylor series method. The efficiency of the method is tested on several numerical examples. It is observed that the method is comparable with the well-known existing methods and in many cases gives better results.

Key words: Newton's method, order of convergence, Maple.

Introduction

Iterative methods for finding the approximate solutions of the nonlinear equations $f(x) = 0$ are being developed using several different techniques (see[4] and the referrences therein). There are many iterative methods for finding a real roots of $f(x) = 0$. One of the well-known methods is the classical Nweton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots$$

This method converges quadratically in some neighbourhood of a simple root a. Some modifications of Nweton' method to achieve higher order convergence rates have been proposed in many papers (Salkuyeh, 2007). We compare the result of our method with the results obtained by other well known methods that are presented in (Abbasbandy, 2003; Chun, 2005; Homeier, 2005). The main proposed of this method is based on an improvement of the Newton-Raphson method.

New Iterative Method and Convergence Analysis

Consider the nonlinear equation $f(x) = 0$, and writing $f(x+h)$ in taylor series expansion about x, we obtain

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + g(h) \tag{1}$$

$$g(h) = f(x+h) - f(x) - hf'(x) - \frac{h^2}{2} f''(x). \tag{2}$$

Supposing $f'(x) \neq 0$, and searches for a value of h such that

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f(x+h) 0 . i.e.

$$f(x) + hf'(x) + \frac{h^2}{2} f''(x) + g(h) = 0. \tag{3}$$

Dividing (3) by *f`(x)* , we get

$$\frac{h^2 f''(x)}{2f'(x)} + h + \frac{f(x) + g(h)}{f'(x)} = 0. \tag{4}$$

Solve (4) for *h*, we get

$$h = -\frac{f'(x)}{f''(x)} + \frac{f'(x)\sqrt{1 - 2\left(\frac{f(x)+g(h)}{f'^2(x)}\right) f''(x)}}{f''(x)}. \tag{5}$$

In (5), we approximate

$$\sqrt{1 - 2\left(\frac{f(x)+g(h)}{f'^2(x)}\right) f''(x)}$$

by using the binomial formula as follows

$$\sqrt{1 - 2\left(\frac{f(x)+g(h)}{f'^2(x)}\right) f''(x)} \approx 1 - \frac{2\big(f(x)+g(h)\big)f''(x)}{2f'^2(x)} - \frac{4\big(f(x)+g(h)\big)f''^2(x)}{8f'^4(x)}. \tag{6}$$

By substitution (2) into (6) and after simplifications, we obtain

$$\sqrt{1 - 2\left(\frac{f(x)+g(h)}{f'^2(x)}\right) f''(x)} = 1 - \frac{\big(2f(x)f'^2(x) + 2f(x+h)f'^2(x) - f^2(x)f''(x)\big)f''(x)}{2f'^4(x)} - \frac{(2f(x)f'^2(x) + 2f(x+h)f'^2(x) - f^2(x)f''(x))^2 f''^2(x)}{8f'^8(x)}. \tag{7}$$

Substituting (7) into (5) yields

$$h = -\frac{f(x)}{f'(x)} - \frac{f(x+h)}{f'(x)} + \frac{f^2(x)f''(x)}{2f'^3(x)} - \frac{(2f(x)f'^2(x) + 2f(x+h)f'^2(x) - f^2(x)f''(x))^2 f''(x)}{8f'^7(x)}. \tag{8}$$

From (8) we have

$$x + h = x - \frac{f(x)}{f'(x)} - \frac{f(x+h)}{f'(x)} + \frac{f^2(x)f''(x)}{2f'^3(x)} -$$

$$\frac{(2f(x)f'^2(x)+2f(x+h)f'^2(x)-f^2(x)f''(x))^2 f''(x)}{8f'^7(x)}. \tag{9}$$

In (9) replace x by x_n, also, in the right-hand side replace $x_n+h=x_{n+1}$ by

$$x_n+h=x_n-\frac{f(x_n)}{f'(x_n)},$$

We obtain the following iterative method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}+\frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}-$$

$$\frac{(2f(x_n)f'^2(x_n)+2f\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)f'^2(x_n)-f^2(x_n)f''(x_n))^2 f''(x_n)}{8f'^7(x_n)}. \tag{10}$$

The iterative method (10) has third order convergence.
Hereunder, we consider the convergence analysis of (10) by the following theorem.

Theorem 1:

Assume that the function $f: I \rightarrow R$ for an open interval I has a simple root $a \in I$. If $f(x)$ is sufficiently smooth on the neighborhood of the root α, then the iterative method defined by (10) converges of order three.

Proof:

Let $e_n = x_n - \alpha$, using Taylor's expansion about a, we can write

$$f(x_n)=f'(\alpha)[e_n+c_2e_n^2+c_3e_n^3+\cdots], \tag{11}$$

$$f'(x_n)=f'(\alpha)[1+2c_2e_n+3c_3e_n^2+\cdots], \tag{12}$$

$$f''(x_n)=f'(\alpha)[2c_2+6c_3e_n+12c_4e_n^2+\cdots], \tag{13}$$

where

$$c_j=\frac{f^j(\alpha)}{j!\,f'(\alpha)}, \text{for } j=2,3,\ldots.$$

Dividing (11) by (12) gives

$$\frac{f(x_n)}{f'(x_n)}=e_n-c_2e_n^2+2(c_2^2-c_3)e_n^3+O(e_n^4). \tag{14}$$

From (14), we get

$$x_n-\frac{f(x_n)}{f'(x_n)}=c_2e_n^2+2(c_2^2-c_3)e_n^3+O(e_n^4). \tag{15}$$

By expanding $f\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)$ about a, and using (15) we have

$$f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) = c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{16}$$

From (16) and (12), we obtain

$$\frac{f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)} = c_2 e_n^2 + 2(2c_3 - 4c_2^2)e_n^3 + O(e_n^4). \tag{17}$$

Now, by (11), (12) and (13) we have

$$\frac{f^2(x) f''(x)}{2f'^3(x)} = c_2 e_n^2 + (3c_3 - 4c_2^2)e_n^3 + O(e_n^4). \tag{18}$$

From (11), (12), (13) and (16) we get

$$k = c_2 e_n^2 + (3c_3 - 4c_2^2)e_n^3 + O(e_n^4), \tag{19}$$

where

$$k = \frac{(2f(x_n)f'^2(x_n) + 2f(y_n)f'^2(x_n) - f^2(x_n)f''(x_n))^2 f''(x_n)}{8f'^7(x_n)}.$$

Substitiuting (14), (17), (18) and (19) into (10), we obtain

$$e_{n+1} = 2c_2^2 e_n^3 + (-9c_2^3 + 9c_2 c_3)e_n^4 + O(e_n^5).$$

This means that the iterative method (10) has third order convergences. Also we find the order convergence of (10) by using Maple as follows:

```
> y:=x->x-f(x)/(D(f)(x));
```

$$y := x \to x - \frac{f(x)}{f'(x)}$$

```
>z:=x->x-f(x)/D(f)(x)-(f@y)(x)/D(f)(x)+f(x)^2*D(D(f))(x)/(2*D(f)(x)^3)-(D(D(f))(x)*(2*D(f)(x)^2*(f@y)(x)+2*D(f)(x)^2*f(x)-f(x)^2*D(D(f))(x))^2)/(8*D(f)(x)^7);
```

$$z := x \to x - \frac{f(x)}{D(f)(x)} - \frac{f@(y)}{D(f)(x)} + \frac{f(x)^2 D(D(f))(x)}{2D(f)(x)^3} - \frac{(2f(x)D(f)(x)^2 + 2f@y(x)D(f)(x)^2 - f^2(x)D(D(f))(x))^2 D(D(f))(x)}{8D(f)(x)^7}$$

```
> algsubs (f (α) =0, w (α));              α
> algsubs (f (α) =0, D (w) (α));          0
> algsubs (f (α) =0,D(D (w)) (α));        0
> algsubs (f (α) =0, (D@@3) (w) (α)));
```

$$\frac{3D^{(2)}(f)(\alpha)^2}{D(f)(\alpha)^2}$$

Thus, we obtain

$$z(\alpha) = \alpha, \quad z'(\alpha) = z^{(2)}(\alpha) = 0,$$

and

$$z^{(3)}(\alpha) = \frac{3D^{(2)}(f)(\alpha)^2}{2D(f)(\alpha)^2}.$$

So, it is of order three.

Numerical Examples

We present some numerical examples to illustrate the efficiency of the iterative method proposed in this section. We compare the Newton's (NM) (1), the method of Abbasbandy (AM)(Abbasbandy, 2003), the method of Homeier (HM), (Homeier, 2005), the method of Chun (CM)(Chun, 2005), and our new method (10) (N1).

All computations we done using Maple 9.5. We accept an approximate solutions rather than the exact root, depending on the precision ε. We use the following stopping criteria are used for computer programs:

$$|f(x_n)| < 10^{-15}$$

$$|x_{n+1} - x_n| < 10^{-15}$$

We used the following test functions and displays the computed approximate zero X_* using 27^{th} decimal place.

$$f_1(x) = cosx - x, \qquad x_* = 0.739085133215606416553120876$$

$$f_2(x) = x^3 + 4x^2 - 10, \quad x_* = 1.365230013414096845760806829$$

$$f_3(x) = sinx - \frac{x}{2}, \qquad x_* = 1.895494267033980947144035738$$

$$f_4(x) = (x+2)e^x - 1, x_* = -0.442854401002388583141328000$$

$$f_5(x) = (x-1)^3 - 1, \quad x_* = 2$$

$$f_6(x) = x^2 - e^x - 3x + 2, x_* = 0.257530285439860760455367304$$

As convergence criterion, it was required that the distance of two consecutive approximations for the zero was less than 10^{-15}. The results presented in Table1 are the number of iterations (IT) to approximate the zero by different methods.

Table 1: shows a comparison between the iterative methods NM, AM, HM, CM and our new method N1

Functions	$x_\circ$	IT				
		NM	AM	HM	CM	N1
f_1	1.7	5	4	4	4	3
f_2	3	6	4	4	4	4
f_3	2.3	5	4	3	5	3
f_4	2	8	6	5	6	6
f_5	3.5	6	4	4	5	5
f_6	2	6	5	5	4	4

Conclusion

In this paper, we suggest and analyze new third-order iterative methods for solving nonlinear equations. We observed from numerical examples that the proposed method have at least equal performance as compared with other methods given in (Abbasbandy, 2003; Chun, 2005; Homeier, 2005).

References

Abbasbandy, S., 2003. Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, Appl. Math. compute., 145: 887-893.

Chun, C., 2005. Iterative methods improving Newton's method by decomposition, method, Comput. Math. Appl., 50: 1559-1568.

Homeier, H.H.H., 2005. On Newton-type methods with cubic convergence, J. Comput. Appl. Math., 176: 425-432.

Noor, M.A., 2010. Some iterative methods for solving nonlinear equations using homotopy oerturbation mehtod, International journal of computer Mathematics, 87(1): 141-149.

Salkuyeh, D.K., 2007. A family of Newton-type methods for solving nonlinear equations, International journal of computer Mathematics, 84(3): 411-419.