On Existence-Uniqueness of Solution to Countable Number of First-Order Differential Equations in the Space \( l_2 \)

Gafurjan Ibragimov and Abbas Badakaya Ja’afaru

Institute for Mathematical Research and Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia

ABSTRACT

We study a model consisting of countable number of first order differential equations with function coefficients. The model arises as a result of use of decomposition method to simplify control and differential game problems described by some partial differential equations. In this paper, we prove existence-uniqueness theorem of solution to the model in Hilbert space \( l_2 \). The result announce in this paper permits the investigation of optimal control and differential games problems described by such a model in the space considered.

Key words: Differential equations, countable number, solution, Hilbert space.

Introduction

Control and differential game problems described by partial differential are of increasing interest because of their significance importance in solving engineering, economics and military operations related problems. Some of the control and differential game problems described by parabolic and hyperbolic partial differential equations can be reduced to those described by countable number of ordinary differential equations using decomposition method (see, for example, Chernous’ko, 1992; Ibragimov, 2003; Satimov and Tukhtasinov, 2006 and Satimov and Tukhtasinov, 2007). In particular, use of decomposition method in Chernous’ko, 1992; Ibragimov, 2003; and Satimov and Tukhtasinov, 2006, in the study of control and differential game problems described by the following partial differential equation

\[
Tu = Au + w, \quad (1)
\]

reduces the problem to the one described by the countable number of ordinary differential equations

\[
\dot{z}_k(t) + \mu z_k(t) = w_k(t), \quad k = 1, 2, ..., \quad (2)
\]

where in (1) \( u = u(t, x) \) is a scalar function of \( x \in \mathbb{R}^n \) and time \( t \); \( w_k \) are control parameters;

\[
Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right)
\]

is a linear differential operator whose coefficients do not depend on \( t \). In (2), \( w_k, k = 1, 2, ... \) are control parameters and constants \( \mu_k, k = 1, 2, ... \) satisfy the condition.

\[ 0 \leq \mu_1 \leq \mu_2 \rightarrow \infty. \]

This shows a significant relationship between the problems described by partial differential equations and those that described by countable number of ordinary differential equations. Therefore, the later can be studied in an independent frame.

The purpose of this paper is paper is to investigate the existence-uniqueness of solution to system (2) with function coefficients in the space \( l_2 \) with a view to investigate differential game problem described by such a system in the space.

Corresponding Author: Gafurjan Ibragimov, Institute for Mathematical Research and Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
E-mails: gafur@science.upm.edu.my
Statement of the Problem:

We introduce the space

\[ l_2 = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\}, \]

with inner product and norm

\[ \langle \alpha, \beta \rangle = \sum_{k=1}^{\infty} \alpha_k \beta_k, \quad \alpha, \beta \in l_2, \quad \| \alpha \| = \left( \sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}. \]

Let

\[ L_2(0, T; l_2) = \left\{ w(t) = (w_1(t), w_2(t), \ldots) : \sum_{k=1}^{\infty} \int_0^T w_k^2(t) dt < \infty, \quad w_k \in L_2(0, T) \right\}, \]

\[ \| w(t) \|_{L_2(0, T; l_2)} = \left( \sum_{k=1}^{\infty} \int_0^T w_k^2(t) dt \right)^{1/2}, \]

where \( T, \ T > 0, \) is a given number.

We consider the system of differential equations

\[ \dot{z}_k(t) + \lambda_k(t) z(t) = w_k(t), \quad z_k(0) = z_k^0, \quad k = 1, 2, \ldots, \tag{3} \]

where \( z_k, w_k \in \mathbb{R}^1, \ k = 1, 2, \ldots, \) \( z_0 = (z_1^0, z_2^0, \ldots) \in l_2, \ w_1, w_2, \ldots, \) are control parameters and \( \lambda_k(t), \ k = 1, 2, \ldots \) is a bounded sequence of arbitrary positive continuous functions such that \( \lambda_k(0) = 0 \) for all \( k. \)

Definition 1: A function \( z(t) = (z_1(t), z_2(t), \ldots), \) \( 0 \leq t \leq T, \) is called the solution of the system (3) if each coordinates \( z_k(t) \) of that

1) is continuous and differentiable on \( (0, T), \) and satisfies the initial conditions \( z_k(0) = z_k^0, \)

2) has the first derivative \( \dot{z}_k(t) \) on \( (0, T) \) and satisfies the equation

\[ \dot{z}_k(t) + \lambda_k(t) z(t) = w_k(t), \quad k = 1, 2, \ldots, \]

almost everywhere on \( (0, T). \)

Main Result:

It is not difficult to verify that the \( kth \) equation in (3) has a unique solution

\[ z_k(t) = e^{-\alpha_k(t)} \left( z_k^0 + \int_0^t w_k(s) e^{\alpha_k(s)} ds \right), \tag{4} \]

where \( \alpha_k(t) = \int_0^t \lambda_k(s) ds. \)

Let \( C(0, T; l_2) \) be the space of continuous functions \( z(t) = (z_1(t), z_2(t), \ldots) \) with values in \( l_2. \)

Theorem: If \( z_0 \in l_2 \) and \( w(t) \in L_2(0, T; l_2), \) then the function \( z(t) = (z_1(t), z_2(t), \ldots), \) where \( z_k(t) \) is defined by (4), belongs to the space \( C(0, T; l_2). \)

Proof. The theorem states two claims: first \( z(t) \in l_2 \) and the second \( z(t), \ 0 \leq t \leq T \) is continuous.

We will show that \( z(t) \in l_2 \) for all \( t \in [0, T], \) Indeed, it follows from (4) that
\[ |z_k(t)|^2 \leq 2 |z_{i_0}|^2 e^{-2\alpha(t)} + 2e^{-2\alpha(t)} \left( \int_0^t w_k(s) e^{\alpha(s)} ds \right)^2, \] 

(here we used the inequality \((a+b)^2 \leq 2(a^2 + b^2)\)).

Using the Cauchy-Schwartz inequality, we have

\[ \left( \int_0^t w_k(s) e^{\alpha(s)} ds \right)^2 \leq \int_0^t w_k^2(s) ds \int_0^t e^{2\alpha(s)} ds, \]

From the fact that \(\lambda_1(t), \lambda_2(t), \ldots\) is a bounded sequence of positive continuous functions, a number \(C\) exists such that

\[ \int_0^t e^{2\alpha(s)} ds \leq C, \]

for all \(k\), since \(t \in [0, T]\). Therefore,

\[ \left( \int_0^t w_k(s) e^{\alpha(s)} ds \right)^2 \leq C \int_0^t w_k^2(s) ds. \] 

Thus (see 5)

\[ |z_k(t)|^2 \leq 2 |z_{i_0}|^2 e^{-2\alpha(t)} + 2Ce^{-2\alpha(t)} \int_0^t w_k^2(s) ds \]

\[ \leq 2 |z_{i_0}|^2 + 2C \int_0^t w_k^2(s) ds, \quad t \in [0, T]. \] 

Summing over \(k\) both sides of the inequality (7), we get

\[ \sum_{k=1}^N |z_k(t)|^2 \leq 2 \sum_{k=1}^N |z_{i_0}|^2 + 2C \sum_{k=1}^N \int_0^t w_k^2(s) ds, \]

and so

\[ \|z(t)\|_2^2 \leq C' \left( \|z_{i_0}\|_2^2 + \|w(t)\|_{L^2(0,T)}^2 \right), \]

where \(C' = \max \{2, 2C\}\). Thus, we have shown that for all \(t \in [0, T]\) the function \(z(t) = (z_1(t), z_2(t), \ldots)\), where \(z_k(t), k = 1, 2, \ldots, \) are defined by (4), satisfies the inclusion \(z(t) \in L^2\).

2. We will check the continuity of \(z(t)\) on the interval \([0, T]\). Consider the expression

\[ \|z_k(t + h) - z_k(t)\|_2^2 \]

\[ = \left( \sum_{k=1}^N \sum_{k=N+1}^N \right) |z_k(t + h) - z_k(t)|^2. \]

The positive numbers \(h\) and \(N\) will be chosen by the given number \(\epsilon > 0\). According to (4)

\[ z_k(t + h) - z_k(t) = z_{i_0}(e^{-\alpha(t+h)} - e^{-\alpha(t)}) + \left(e^{-\alpha(t+h)} - e^{-\alpha(t)}\right) \int_0^t w_k(s) e^{\alpha(s)} ds. \]
+e^{-z_k(t+h)} \int_{t}^{t+h} w_k(s)e^{z_k(s)} \, ds.

Using the inequality \((a+b+c)^2 \leq 3(a^2 + b^2 + c^2)\), we have

\[
|z_k(t+h) - z_k(t)|^2 \leq 3 |z_{k0}|^2 (e^{z_k(t+h)} - e^{z_k(t)})^2 + 3(e^{z_k(t+h)} - e^{z_k(t)})^2 \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2
+3e^{-z_k(t+h)} \int_{t}^{t+h} w_k(s)e^{z_k(s)} \, ds \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2.
\]

Therefore

\[
\|z_k(t+h) - z_k(t)\|_2^2 \leq 3 \sum_{k=1}^{\infty} |z_{k0}|^2 (e^{-z_k(t+h)} - e^{-z_k(t)})^2
+3 \sum_{k=1}^{\infty} (e^{-z_k(t+h)} - e^{-z_k(t)})^2 \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2
+3 \sum_{k=1}^{\infty} e^{-z_k(t+h)} \int_{t}^{t+h} w_k(s)e^{z_k(s)} \, ds \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2.
\]

We now have

\[
\|z(t+h) - z(t)\|_2^2 \leq I_1 + I_2 + I_3,
\]

where \(I_i, i=1,2,3\), are defined as follows:

\[
I_1 = 3 \sum_{k=1}^{\infty} |z_{k0}|^2 \left( e^{-z_k(t+h)} - e^{-z_k(t)} \right)^2 \left[ |z_{k0}|^2 + \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2 \right],
\]

\[
I_2 = 3 \sum_{k=1}^{\infty} (e^{-z_k(t+h)} - e^{-z_k(t)})^2 \left[ |z_{k0}|^2 + \left[ \int_{0}^{h} w_k(s)e^{z_k(s)} \, ds \right]^2 \right],
\]

\[
I_3 = \sum_{k=1}^{\infty} e^{-z_k(t+h)} \int_{t}^{t+h} w_k(s)^2 \, ds \int_{t}^{t+h} e^{z_k(s)} \, ds.
\]

From the fact (see definition of norms)

\[
\sum_{k=1}^{\infty} |z_{k0}|^2 < \infty, \quad \sum_{k=1}^{\infty} \left[ \int_{0}^{h} w_k(s) \, ds \right]^2 < \infty,
\]

with (6) and the inequality

\[
(e^{-z_k(t+h)} - e^{-z_k(t)})^2 \leq e^{-2z_k(t+h)} + e^{-2z_k(t)} \leq 2,
\]

we have, for any number \( \varepsilon > 0 \), there exists a positive number \( N \) such that \( I_1 < \frac{\varepsilon}{4} \). After that, from the fact that \( I_1 \) consists of finite number of summands and that \( \left( e^{-z_k(t+h)} - e^{-z_k(t)} \right)^2 \to 0 \) as \( h \to 0 \), for any \( \varepsilon > 0 \), there exists \( \delta_1 \) such that \( I_1 < \frac{\varepsilon}{4} \) whenever \( 0 \leq h < \delta_1 \).

For \( I_2 \), using (6) we have

\[
I_2 \leq C \sum_{k=1}^{\infty} e^{-2z_k(t+h)} \int_{t}^{t+h} w_k(s)^2 \, ds.
\]

Since \( \lambda_1(t), \lambda_2(t), \ldots \), is a bounded sequence of positive continuous functions, then \( e^{-2z_k(t+h)} \leq 1 \),

for all \( k \). Therefore, by
\[
\sum_{k=1}^{N} \left[ \int_{t}^{t+h} |w_k(s)|^2 \, ds \right] \leq \frac{\varepsilon}{6}
\]

for any \( \varepsilon > 0 \) as \( h \rightarrow 0 \). The numbers \( \delta_1, \delta_2 > 0 \) are chosen such that

\[
\sum_{k=1}^{N} \left[ \int_{t}^{t+h} |w_k(s)|^2 \, ds \right] \leq \frac{\varepsilon}{6},
\]

whenever \( 0 \leq h < \delta_1 \). For such \( h \) we can take \( l_1 \leq \varepsilon / 3 \) whenever \( 0 \leq h < \delta_1 \).

Thus, \( \| z(t+h) - z(t) \|_{L_2} \) can be done less than any given \( \varepsilon > 0 \), by choosing \( N \) and \( \delta = \min \{ \delta_1, \delta_2 \} \).

Similarly, we can consider \( \| z(t) - z(t-h) \|_{L_2} \), \( h > 0 \) and show that for any given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \| z(t) - z(t-h) \|_{L_2} < \varepsilon \) whenever \( 0 \leq h < \delta_2 \). Hence, \( z(\cdot) \in C(0,T; l_2) \). The proof of the theorem is complete.

References


