**ORIGINAL ARTICLES**

**On Solvability of Fuzzy System of Linear Matrix Equations**

N. Mikaeilvand

*Department of Mathematics, Islamic Azad University- Ardabil Branch, Ardabil, Iran.*

**ABSTRACT**

System of equations is the simplest and the most useful mathematical model for a lot of problems considered by applied mathematics. The main aim of this paper is to illustrate and discuss Fuzzy System of Linear Matrix Equations (shown as FLME) of the form $A_1XB_1 + A_2XB_2 + \ldots + A_lXB_l = C$ for finding its fuzzy number solutions. The parametric form of fuzzy numbers is used. We use embedding method to transform fuzzy system of linear matrix equations to parametric system of linear equations. Necessary and sufficient conditions for existence of fuzzy number solutions in special cases are derived and numerical procedure for calculating the solutions is designed.

**Key words:** Fuzzy System of Linear Matrix Equations, Embedding method, Conditional inverse, Fuzzy system of linear equation, Kronecker matrix product, Fuzzy Lyapunov equation.

**Introduction**

Let $R^{m \times n}$ denote the vector space of real matrices of order $m \times n$ and $F^{m \times n}$ denote the set of all $m \times n$ matrices which its elements are fuzzy numbers. For given matrices $A_1, \ldots, A_l \in R^{m \times n}$, $B_1, \ldots, B_l \in R^{n \times q}$ and $C \in F^{n \times r}$, we consider the problem of determining the solutions $X$ of the matrix equation

$$A_1XB_1 + A_2XB_2 + \ldots + A_lXB_l = C.$$ (1)

A special case of this equation where $l = 2 (A_1XB_1 + A_2XB_2 = C)$ occur in MINQUE theory of estimating covariance components in a covariance components model Shu-Xin, (2008) and are likely to occur elsewhere and $B_1$ and $A_2$ are identity matrices is transformed to $A_1X + XB_2 = C$. A familiar example occurs in the Lyapunov theory of stability[18] with $B = A^t$. It also appear in many different engineering and mathematical perspectives such as control theory, system theory, optimization, power systems, signal processing, linear algebra, differential equations, boundary value problems, large space flexible structures, and communications (Dubois and Prade 1980; Friedman *et al.*, 2003).

Research on solving systems of linear matrix equations has been ongoing for the past 50 or more years. When $l=1$ and $B \in R^{m \times n}$, fuzzy linear matrix equation transforms to a fuzzy linear equation that has been studied by many authors (Allahviranloo *et al.*, 2009; 2005; 2004; 2005; 2003; 2006; 2006; 2007; 2005; 2009; 2006; Friedman *et al.*, 1998; 2003; Ma *et al.*, 2000; Shu-Xin, 2008; Ke Wang, 2006; 2006; 2006; Xizhao Wang, 2001; Bing Zheng, 2006).

**Corresponding Author:** N. Mikaeilvand, Department of Mathematics, Islamic Azad University- Ardabil Branch, Ardabil, Iran.  
Email: mikaeilvand@aol.com
Fuzzy linear matrix equation is introduced by Allahviranloo et al. (2009). In this paper, their method is extended for solving (1). The embedding method is used and replace the original fuzzy linear matrix approach by parametric - crisp function - linear matrix equations.

This paper is organized as follows:

In section 2, we discuss some basic definitions and results on fuzzy numbers and the fuzzy linear matrix equations FLME. In section 3, The proposed model for solving the linear matrix equation and its limitations are discussed and it is illustrated by solving some numerical examples. The conclusion is drawn in section 4.

**Background:**

An arbitrary fuzzy number $\tilde{u}$ is represented by an ordered pair of functions $(u(r), \bar{u}(r))$; $0 \leq r \leq 1$ which satisfy the following requirements Goetschel and Voxman, (1986):

- $u(r)$ is a bounded monotonic increasing left continuous function;
- $\bar{u}(r)$ is a bounded monotonic decreasing left continuous function;
- $u(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

A crisp number $k$ is simply represented by $\tilde{u}(r) = u(r) = k$; $0 \leq r \leq 1$, and called singleton. The set of all fuzzy numbers is denoted by $E$.

For arbitrary $\tilde{u} = (u(r), \bar{u}(r))$, $\tilde{v} = (v(r), \bar{v}(r))$ and scalar $k$ we define addition $(\tilde{u} + \tilde{v})$ and scalar multiplication by $k$ as

**Addition:**

\[
(u + v)(r) = u(r) + v(r),
\]

\[
(\tilde{u} + \tilde{v})(r) = \bar{u}(r) + \bar{v}(r),
\]

**Scalar Multiplication:**

\[
k\tilde{u} = \begin{cases} 
(ku(r), k\bar{u}(r)), & k \geq 0, \\
(k\bar{u}(r), k\bar{u}(r)), & k < 0.
\end{cases}
\]

For two arbitrary fuzzy numbers $\tilde{x} = (x(r), \bar{x}(r))$ and $\tilde{y} = (y(r), \bar{y}(r))$, $\tilde{x} = \tilde{y}$ if and only if $x(r) = y(r)$ and $\bar{x}(r) = \bar{y}(r)$.

**Definition 2.1**

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times q$ matrices, respectively, the Kronecker product

\[
A \otimes B = (a_{ij}B) \quad \text{is a } mp \times nq \text{ matrix expressible as a partitioned matrix with } a_{ij}B \text{ as the } (i,j) \text{ th partition, } i = 1, \cdots , m; j = 1, \cdots , n.
\]

**Fuzzy System Of Linear Matrix Equations:**

**Definition**

The linear matrix equation (1) is called a fuzzy linear matrix equations, FLME, if the left coefficient matrices $A_i = (a_{ij})$ ($1 \leq i \leq m, 1 \leq j \leq n, 1 \leq r \leq l$) and the right coefficient matrices ...
\( B_r = (b_{rj}) \) \((1 \leq i \leq p, 1 \leq j \leq q, 1 \leq r \leq l)\) are crisp matrices and the right-hand side matrix \( C = (\tilde{c}_y) \) \((1 \leq i \leq m, 1 \leq j \leq q)\) is a fuzzy number matrix.

The \( ij \)-th equation of this system is:

\[
\sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} = \tilde{c}_y, \quad 1 \leq i \leq m, \quad 1 \leq j \leq q.
\]

(4)

When \( k=1 \), (4) is reduced to

\[
\sum_{i=1}^{p} a_{ik} x_{ki} b_{ij} = \tilde{c}_y, \quad 1 \leq i \leq m, \quad 1 \leq j \leq q.
\]

(5)

A basic method to express (1) in an equivalent vector form as follows. The column string of \( X(x) \) is the column vector obtained by writing the column of \( X \) one below the other in the natural order. Let \( \Gamma \) denote the matrix of order \( mq \times np \)

\[
\Gamma = B_1^T \otimes A_1 + B_1^T \otimes A_2 + \ldots + B_1^T \otimes A_i
\]

(6)

Where \( \otimes \) denotes Kronecker product Rao and Mitra, (1971), then (1) is equivalent to the equation

\[
\Gamma x = c.
\]

(7)

Definition:

A fuzzy number matrix \( X = (x_{ij}) \) \((1 \leq i \leq m, 1 \leq j \leq q)\) given by \( x_{ij} = (x_{ij}(r), \overline{x}_{ij}(r)) \) \((1 \leq i \leq m, 1 \leq j \leq q)\) is called a solution of the fuzzy linear matrix equation FMLE if:

\[
\sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \tilde{c}_y (r),
\]

(8)

\[
\sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \overline{\tilde{c}_y} (r).
\]

In particular, if \( A_1, \ldots, A_k \) and \( B_1, \ldots, B_k \) are nonnegative matrices, simultaneously, we simply get

\[
\sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \tilde{c}_y (r),
\]

(9)

\[
\sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \sum_{r=1}^{l} \sum_{i=1}^{p} \sum_{k=1}^{n} a_{rik} x_{ki} b_{rj} (r) = \overline{\tilde{c}_y} (r).
\]

In general, however, an arbitrary equation for either \( \tilde{c}_y (r) \) or \( \overline{\tilde{c}_y} (r) \) may include a linear combination of \( x_{ij}(r) \)’s and \( \overline{x}_{ij}(r) \)’s. Consequently, in order to solve the system given by (9), one must solve a crisp linear system where the right hand side vector is the function vector \( (\tilde{c}_{i1}, \ldots, \tilde{c}_{in}, \overline{\tilde{c}}_{i1}, \ldots, \overline{\tilde{c}}_{in}) \).

Let us now rearrange the linear matrix equation \( \Gamma x = c \) such that the unknowns are \( x_y, \overline{x}_y \) \((1 \leq i \leq n, 1 \leq j \leq p)\).
Assume the $2mq \times 2np$ matrix $S=(s_{ij})$ is determined as follows:

$$
\Gamma_{ij} \geq 0 \rightarrow s_{ik,j} = s_{ik,v,k+np} = \Gamma_{ij}
$$

$$
\Gamma_{ij} \leq 0 \rightarrow s_{ik, np+ij} = s_{ik,v,k+np} = -\Gamma_{ij}
$$

and any $S_{ij}$ which is not determined by (10) is zero. Using the matrix notation, the system (9) is extended to the following crisp block form

$$
S = Y
$$

where $S = (s_{ij})$ $(1 \leq i \leq 2mq, \ 1 \leq j \leq 2np)$ and

$$
X = \begin{pmatrix} X \cr \bar{X} \end{pmatrix} = \begin{pmatrix} x_{i1}(r) \\
\vdots \\
x_{imq}(r) \\
-\bar{x}_{i1}(r) \\
\vdots \\
-\bar{x}_{imq}(r) \end{pmatrix},
$$

$$
Y = \begin{pmatrix} Y \cr \bar{Y} \end{pmatrix} = \begin{pmatrix} \xi_{i1}(r) \\
\vdots \\
\xi_{imq}(r) \\
-c_{i1}(r) \\
\vdots \\
-c_{imq}(r) \end{pmatrix},
$$

where $\Xi = SY$ is one solution vector of $S = Y$.

The structure of $S$ implies that $s_{ij} \geq 0 \ (1 \leq i \leq 2mq, \ 1 \leq j \leq 2np)$ and that

$$
S = \begin{pmatrix} E & F \\
F & E \end{pmatrix},
$$

where $E$ contains the positive entries of $\Gamma$, and $F$ contains the absolute values of the negative entries of $\Gamma$, and $\Gamma = E - F$, which implies that the rest of the entries are zero.
Example 3.1:

Consider the fuzzy linear matrix \( A_1XB_1 + A_2XB_2 = C \) where

\[
A_1 = \begin{pmatrix}
1 & 2 & -3 \\
-1 & 0 & 2
\end{pmatrix},
\]

\[
B_1 = \begin{pmatrix}
-1 & 1 \\
1 & -2 \\
-3 & -2
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
-1 & 3 & -2 \\
2 & 1 & 1
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
-1 & 0 \\
-2 & 2 \\
3 & -2
\end{pmatrix},
\]

and

\[
C = \begin{pmatrix}
(33r - 25, -34r + 42) & (36r - 37, -36 + 35) \\
(15r - 14, -17r + 18) & (19r - 20, -15r + 14)
\end{pmatrix}.
\]

Using the Kronecker product, we can write the above equations in the form \( \mathbf{\Gamma} \mathbf{x} = \mathbf{e} \) where

\[
\mathbf{\Gamma} = \begin{pmatrix}
0 & 3 & -6 & -5 & -4 & 3 & 5 & 1 & 3 \\
1 & -4 & 0 & 2 & 2 & -10 & -3 & 2 & 10 \\
-3 & -3 & 3 & -1 & -2 & 3 & -3 & 0 & -3 \\
1 & 2 & -6 & 0 & 2 & -2 & 2 & -2 & -6
\end{pmatrix},
\]

\[
\mathbf{x} = \begin{pmatrix}
x_{11} \\
x_{21} \\
x_{31} \\
\vdots \\
x_{13} \\
x_{23} \\
x_{33}
\end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix}
33r - 25 \\
15r - 14 \\
36r - 37 \\
19r - 20 \\
-34r + 42 \\
-17r + 18 \\
-36r + 35 \\
-15r + 14
\end{pmatrix}.
\]

i.e.
\[
S = \begin{pmatrix}
E & F \\
F & E
\end{pmatrix},
\]

where

\[
E = \begin{pmatrix}
0 & 3 & 0 & 0 & 0 & 3 & 5 & 1 & 3 \\
1 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 10 \\
0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0
\end{pmatrix},
\]

\[
F = \begin{pmatrix}
0 & 0 & 5 & 4 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 10 & 3 & 0 & 0 \\
3 & 3 & 0 & 1 & 2 & 0 & 3 & 0 & 3 \\
0 & 0 & 6 & 0 & 0 & 2 & 0 & 2 & 6
\end{pmatrix}.
\]

When \( k = 1 \), the \( 2me \times 2nr \) matrix \( S = (S_{ij}) \) is determined as follows:

\[\begin{align*}
a_{ij} \geq 0 \text{ and } b_{ik} \geq 0 & \text{ or } a_{ij} \leq 0 \text{ and } b_{ik} \leq 0 \\
\rightarrow s_{ik,j} &= s_{\text{me+ik},nr+ij} = a_{ij} b_{ik}
\end{align*}\]  

\[\begin{align*}
a_{ij} \geq 0 \text{ and } b_{ik} \leq 0 & \text{ or } a_{ij} \leq 0 \text{ and } b_{ik} \geq 0 \\
\rightarrow s_{ik,ar+ij} &= s_{\text{me+ik},j} = -a_{ij} b_{ik},
\end{align*}\]  

and \( E \) contains the positive entries of \( A \otimes B \), and \( F \) contains the absolute values of the negative entries of \( A \otimes B \), and \( A \otimes B = E - F \), which implies that the rest of the entries are zero.

If \( A_1 \) and \( B_1 \) contain the positive entries of \( A \) and \( B \), respectively, and \( A_2 \) and \( B_2 \) contain the negative entries of \( A \) and \( B \) respectively, it is obvious that \( A = A_1 - A_2, \ B = B_1 - B_2 \) and

\[
E = A_1 \otimes B_1^t + A_2 \otimes B_2^t, \\
F = A_2 \otimes B_1^t + A_1 \otimes B_2^t.
\]

When \( mq = np \), the linear system (11) can be solved if and only if matrix \( S \) is nonsingular.

The linear matrix equation \( AXB = C \) is now a general crisp function system of linear equations and can be solved for \( X \). Therefore, we must answer this question: Does the crisp function linear system have a solution when the fuzzy linear matrix equation has a solution?

The next example shows that \( SX = Y \) may have no solution or an infinite number of solutions even if \( AXB = C \) has a unique solution.

**Example 3.2:**

Consider the fuzzy linear matrix \( AXB = C \) where

\[
A = \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
A\otimes B' is nonsingular and hence \((A\otimes B')x=c\) has a unique solution, whereas,

\[
S = \begin{pmatrix}
E & F \\
F & E
\end{pmatrix},
\]

is singular, where

\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}, \quad F = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

In other words, a fuzzy linear matrix equation may have no solution or an infinite number of solutions. Friedman et al. (1998) proved, for square matrices, that \(S\) is nonsingular if and only if matrices \(E-F\) and \(E+F\) are both nonsingular. In particular:

**Theorem 3.1**

If \(A\) and \(B\) are square matrices, \(S\) is nonsingular if and only if matrices \(A, B, A + A_1, B_1 + B_2\) are nonsingular.

**Proof** By adding the \((mq+i)\) th row of \(S\) to its \(i\) th row for \(1 \leq i \leq mq\) we obtain

\[
S = \begin{pmatrix}
E & F \\
F & E
\end{pmatrix} \rightarrow \begin{pmatrix}
E+F & E+F \\
F & E
\end{pmatrix} = S_1.
\]

Then, we subtract the \(j\) th column of \(S_1\) from its \((mq+j)\)-th column for \(1 \leq j \leq mq\) and obtain

\[
S_1 = \begin{pmatrix}
E+F & E+F \\
E & E-F
\end{pmatrix} \rightarrow \begin{pmatrix}
E+F & 0 \\
E & E-F
\end{pmatrix} = S_2.
\]

Clearly, \(|S|=|S_1|=|S_2|=|E+F||E-F|\). Therefore, \(|S|\neq 0\) if and only if \(|E+F|\neq 0\) and \(|E-F|\neq 0\).

Since \(E-F = A \otimes B'\) and \(E+F = (A_1 + A_2) \otimes (B_1 + B_2)'\), Then \(|S|\neq 0\) if and only if \(|A \otimes B'|\neq 0\) and \(|(A_1 + A_2) \otimes (B_1 + B_2)'|\neq 0\).

\(|A \otimes B'|\neq 0\) if and only if \(A \otimes B'\) is invertible. \(|(A_1 + A_2) \otimes (B_1 + B_2)'|\neq 0\) if and only if \((A_1 + A_2) \otimes (B_1 + B_2)'\) is invertible.

Since \(A\) and \(B\) are square matrices, \((A \otimes B')^{-1} = A^{-1} \otimes B'^{-1}\), and

\(((A_1 + A_2) \otimes (B_1 + B_2)')^{-1} = (A_1 + A_2)^{-1} \otimes (B_1 + B_2)^{-1}\), then \((A \otimes B')^{-1}\) exists if and only if \(A\) and \(B\) exist and also \(((A_1 + A_2) \otimes (B_1 + B_2)')^{-1}\) exists if and only if \((A_1 + A_2)^{-1}\) and \((B_1 + B_2)^{-1}\) exist.

Therefore, \(|A \otimes B'|\neq 0\) and \(|(A_1 + A_2) \otimes (B_1 + B_2)'|\neq 0\) if and only if \(|A|\neq 0\) and \(|B|\neq 0\). Hence, \(|S|\neq 0\) if and only if \(|A|\neq 0, |B|\neq 0, |A_1 + A_2|\neq 0\) and \(|B_1 + B_2|\neq 0\), which concludes the proof.
Corollary 3.1

When \( mq= np \), if a crisp linear system does not have a unique solution, the associated fuzzy matrix linear system does not have one either.

To illustrate the solution to a fuzzy linear matrix equation, we first discuss the generalized inverses of matrix \( S \) in a special structure.

**Theorem 3.2**

[6, 28] Let matrix \( S \) be in the form introduced in ( ), then the matrix

\[
S^* = \frac{1}{2} \left( (E+F)^\dagger + (E-F)^\dagger - (E+F)^\dagger (E-F)^\dagger 
- (E-F)^\dagger (E+F)^\dagger + (E-F)^\dagger \right)
\]

is a \( g \)-inverses of matrix \( S \), where \((E+F)^\dagger \) and \((E-F)^\dagger \) are \( g \)-inverses of matrices \((E+F)\) and \((E-F)\), respectively.

In particular, the Moore-Penrose inverse of matrix \( S \) is

\[
S^+ = \frac{1}{2} \left( (E+F)^\dagger + (E-F)^\dagger - (E+F)^\dagger (E-F)^\dagger 
- (E-F)^\dagger (E+F)^\dagger + (E-F)^\dagger \right).
\]

Since \( E-F = A \otimes B^\dagger \) and \( (A \otimes B^\dagger)^\dagger = A^\dagger \otimes B^\dagger \), we obtain the following corollary.

**Corollary 3.2**

Let matrix \( S \) be in the form introduced in ( ), then the matrix

\[
S^* = \frac{1}{2} \begin{pmatrix} a & b \\ b & a \end{pmatrix}
\]

Where

\[
a = (A_1 + A_2)^\dagger \otimes (B_1 + B_2)^\dagger + A^\dagger \otimes B^\dagger
\]

\[
b = (A_1 + A_2)^\dagger \otimes (B_1 + B_2)^\dagger - A^\dagger \otimes B^\dagger
\]

is a \( g \)-inverse of the matrix \( S \). In particular, the Moore-Penrose inverse of matrix \( S \) is

\[
S^+ = \frac{1}{2} \begin{pmatrix} c & d \\ d & c \end{pmatrix}.
\]

Where

\[
c = (A_1 + A_2)^\dagger \otimes (B_1 + B_2)^\dagger + A^\dagger \otimes B^\dagger
\]

\[
d = (A_1 + A_2)^\dagger \otimes (B_1 + B_2)^\dagger - A^\dagger \otimes B^\dagger
\]

Using the above result, we provide the necessary and sufficient condition for the existence of the solution to the system \( SX = Y \).
Theorem 3.3

A necessary and sufficient condition for $S \otimes Y$ to have a solution is that $(A \otimes B')x = Z$ and

$$((A_1 + A_2)^\top \otimes (B_1 + B_2)^\top)x = V$$

should have a solution, where $V = Y - \bar{Y}$ and $Z = Y + \bar{Y}$.

proof A necessary and sufficient condition for $SX=Y$ to be consistent is that $SS^*Y = Y$.

Observe that $SS^*Y = Y$ if and only if

$$\begin{align*}
\frac{1}{2} &\left[ (A_1 + A_2)^\top \otimes (B_1 + B_2)^\top \right] ((A_1 + A_2) \otimes (B_1 + B_2)^\top) (Y - \bar{Y})
+ (A \otimes B') \left( A \otimes B' \right) (Y + \bar{Y}) = Y \\
\frac{1}{2} &\left[ (A_1 + A_2)^\top \otimes (B_1 + B_2)^\top \right] ((A_1 + A_2) \otimes (B_1 + B_2)^\top) (Y - \bar{Y})
- (A \otimes B') \left( A \otimes B' \right) (Y + \bar{Y}) = -\bar{Y}
\end{align*}$$

if and only if

$$\begin{align*}
\left[ (A_1 + A_2)^\top \otimes (B_1 + B_2)^\top \right] ((A_1 + A_2) \otimes (B_1 + B_2)^\top) (Y - \bar{Y})
(A \otimes B')(A \otimes B')(Y + \bar{Y}) = (Y + \bar{Y})
\end{align*}$$

if and only if

$$\begin{align*}
((A_1 + A_2) \otimes (B_1 + B_2)^\top)x &= V \\
(A \otimes B')x &= Z
\end{align*}$$

where $V = Y - \bar{Y}$ and $Z = Y + \bar{Y}$ are consistent.

So far, we have determined when the parametric linear system $S \otimes Y$ has a solution. Since $Y$ and $\bar{Y}$ are linear combinations of $\underline{Y}$ and $\bar{Y}$, then $Y$ and $\bar{Y}$ are bounded and left continuous.

The following results provide a sufficient condition for one solution vector of (13) to be a fuzzy solution vector of (1), which in fact answers the above three questions.

Theorem 3.4

[5] The solution $\square = S \otimes Y$ of (13) is a fuzzy vector for arbitrary $Y$ if $S'$ is nonnegative.

Since the g-inverse of matrix $S$ is not unique, our suggested g-inverse of this matrix might not be nonnegative. Hence, we will give some results for such an $S'$ and $S''$ to be nonnegative.

Theorem 3.5

[28] $S$ admits a nonnegative g-inverse if and only if $S$ has a $\{2\}$ – inverse of the form

$$\begin{pmatrix}
D_1F'D_1 & D_2F'D_1 \\
D_3F'D_1 & D_4F'D_1
\end{pmatrix}$$

where $D_1, D_2, D_3$ and $D_4$ are nonnegative diagonal matrices.

Any matrix admits a unique Moore - Penrose inverse.
Theorem 3.6

Bing Zheng, (2006) $S^+ \geq 0$ if and only if $S^+ = \begin{pmatrix} DE^+ & DF^+ \\ DF^+ & DE^+ \end{pmatrix}$

for some positive diagonal matrix D. In this case, $(E+F)' = D(E+F)'$, $(E-F)' = D(E-F)'

If the given $2mp \times 2nq$ crisp function system $SX=Y$, does not have such a g-inverse $S^{-} \geq 0$, one can always find some vector Y for which the solution $X$ is not a fuzzy vector. In this case, any condition which guarantees a fuzzy solution vector must depend on Y as well as on S.

Now, we define the general fuzzy solution vector of the system, regarding the following theorem.

Theorem 3.7

Rao and Mitra (1971) A general solution of a consistent equations $S\vec{X} = \vec{Y}$ is $\vec{X} = S^{-}\vec{Y} + (I - H)Z$ where $H = S^{-}S$ and Z is an arbitrary vector.

Now, we define general fuzzy number solution of original linear matrix equations

Definition 3.3

Let U be a fuzzy matrix defined by definition(3.2). Then the general fuzzy solution of $AXB = C$ is

$$X = U^+ A^- AZB^- B^-$$

where $Z$ is an arbitrary fuzzy matrix.

Example 3.3

Consider the fuzzy linear matrix equation $AXB = C$ where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} (1+r, 3-r) & (-1+2r, 4-3r) \\ (7, 8-r) & (-2+r, -r) \end{pmatrix}.$$

For solving this system, first we transform it to $(A \otimes B')\vec{x} = \vec{c}$, where

$$A \otimes B' = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

The extended $8 \times 4$ matrix is
By simple calculation, the Moore-Penrose inverse of $S$ is

$$S^+ = \begin{bmatrix}
\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4}
\end{bmatrix} \geq 0$$

and hence

$$x_1 = (-1 + r, \frac{3}{4} - \frac{3}{4} r),$$

$$x_2 = (-\frac{3}{4} - \frac{3}{4} r, 1 - r).$$

Here, $x_2 \leq x_1, x_2 \leq x_2$ and $x_1, x_2$ are monotonically increasing functions and $x_1, x_2$ are monotonically decreasing functions. Therefore, this solution can define fuzzy solution.

**Example 3.4**

Consider the fuzzy linear matrix equation $AXB = C$ where

$$A = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} (r, 2 - r) \\ (2r, 3 - r) \end{bmatrix}$$

For solving this system, first it is transformed to $(A \otimes B')x = c$, where

$$A \otimes B' = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}.$$

The extended $4 \times 4$ matrix $S$ is

$$S = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}$$
By simple calculation, the Moore-Penrose inverse of $S$ is

$$S^+ = \begin{pmatrix} 0.4 & 0 & 0 & 0.4 \\ 0 & 0.2 & 0.4 & 0 \\ 0.4 & 0 & 0 & 0.4 \\ 0 & 0.2 & 0.4 & 0 \end{pmatrix} \geq 0$$

and hence

$$x_1 = (0.8 - 1.2r, 1.2 - .8r),$$

$$x_2 = (.8r - .8, .8 - 8r).$$

Here, $x_1 \leq x_1, x_2 \leq x_2$ and $x_1, x_2$ are monotonically increasing functions and $\bar{x}_1, \bar{x}_2$ are monotonically decreasing functions. Therefore, this solution can define fuzzy solution. We can define the general set of fuzzy solutions of this system of the form:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (0.8 - 1.2r, 1.2 - 0.8r) \\ (0.8r - 0.8, 0.8 - 0.8r) \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - A^T A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} B^T B.$$

where $z_1, z_2$ are fuzzy numbers.

Conclusion:

Linear matrix equation has many applications in various areas of science (Lancaster and Tismenetsky 1985; Shu-Xin 2008). In this paper, the fuzzy linear matrix equation of the form $A_1 X B_1 + ... + A_n X B_n = C$ and specially, $AXB = C$ is introduced. We found a fuzzy solution of fuzzy linear matrix equations by an analytic approach. For this end, the original system $AXB = C$ is transformed to a parametric system using the Kronecker product and the embedding approach. The necessary and sufficient condition for defining and existence of the fuzzy solution of the original system by solving parametric system was discussed, and the strong and weak fuzzy solutions of original system were defined. Finally, the general set of fuzzy solutions was defined.

Acknowledgment

This paper was essential supported by research grant from Islamic Azad University- Ardabil branch, Ardabil, Iran.

References


