Homotopy Perturbation Method Approach for Solution of Equation to Unsteady Flow of a Polytropic Gas

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Abstract: Aim of the paper is to investigate solution of a non-linear physical equation by Homotopy Perturbation Method (HPM). The model of interest in Physics is considered and solved by Homotopy Perturbation Method. The behavior of Homotopy solutions and the effects of different values of time are investigated. The pertinent features of the technique including numerical illustration of nonlinear physical model have been discussed.

Key words: Differential equations; Homotopy Perturbation Method; Polytropic gas

INTRODUCTION

Nonlinear phenomena appear in many areas of science and engineering such as solid state physics, plasma physics, fluid dynamics, mathematical biology, chemical kinetics etc. are modeled by differential equations. A broad class of analytical methods and numerical methods were used to handle such problems.

Various methods have been proposed such as Finite Difference Method [21,2], Adomian Decomposition Method [5,18-20], Variational Iteration Method [13-16], Integral transform [11] etc.but all these methods have some limitations.

The Homotopy Perturbation Method (HPM) was first proposed by Chinese Mathematicians J.H.He [22,12,6-9,17]. The essential idea of this method is to introduce a homotopy parameter $p$, say which takes the values from 0 to 1 .When $p =0$,the system of equations usually reduces to a simplified form ,which normally admit a rather simple solution .As $p$ gradually increases to 1, the system goes through a sequence of deformation and the solution of each of which is close to that at the previous stage of deformation .Eventually at $p =1$, the system takes the original form of equation and final stage of deformation gives the desired result. One of the most remarkable feature of the HPM is that usually only a few perturbation terms are sufficient to obtain a reasonably accurate solution.

The HPM has been employed to solve a large variety of linear and nonlinear problems. This technique was used by He [9] to find solution of nonlinear boundary value problems. Blasius differential equation was solved by He [7] using Perturbation technique.


Abbasbandy [24] employed the He’s homotopy perturbation technique to solve functional integral equations and the results obtained by Lagrange interpolation formula and HPM were compared.

Ganji and Sadighi [25] considered the nonlinear coupled system of reaction-diffusion equations using the HPM. Ganji and Sadighi [25] reported that the Homotopy Perturbation Technique is a very powerful and efficient scheme to find analytical solutions for a wide class of nonlinear engineering problems and presents a rapid convergence for the solutions. The solutions obtained by HPM show that the results are in excellent agreement with those obtained by Adomian Decomposition Method. A comparison between the HPM and the decomposition procedure of Adomian shows that the former is more effective than the latter, as the HPM can overcome the difficulties arising in calculating Adomian polynomials.

The Homotopy Perturbation Method provides highly accurate solution for nonlinear problems in comparison with numerical techniques. It does not require large computer memory and discretization of variables. It can give the solutions for each point within the domain of interest, unlike the numerical solutions which are available for a particular run, only for a set of discrete points in the domain. The HPM avoids linearization and physically unrealistic assumptions.
Aim of the paper is to investigate solution of governing equations of unsteady flow of a polytropic gas using Homotopy Perturbation Method (HPM).

2. Analysis of Homotopy Perturbation Method: Consider the following nonlinear differential equation

\[ A(w) - f(r) = 0, \quad r \in D, \]  

(1)

with the boundary conditions

\[ B\left(w, \frac{\partial w}{\partial n}\right) = 0, \quad r \in \gamma, \]  

(2)

where A is a general differential operator, B is a boundary operator, f(r) is known analytical function, \( \gamma \) is the boundary of the domain D.

The operator A can generally be divided into two parts L and N, where L is linear and N is nonlinear, therefore eq. (1) can be written as

\[ L(w) + N(w) - f(r) = 0. \]  

(3)

By using Homotopy technique, one can construct a homotopy \( u(r, p) \) : \( D \times [0,1] \rightarrow R \), which satisfies

\[ H(u,p) = (1-p) [L(w) - L(w_0)] + p[A(u) f(r)] = 0, \]  

(4)

where \( p \in [0,1] \) is an embedding parameter and \( w_0 \) is the initial approximation of eq (1) which satisfies the boundary conditions. Obviously, we get

\[ H(u,0) = L(u)L(w_0) = 0, \quad H(u,1) = A(u)f(r) = 0. \]  

(5)

The changing process of \( p' \) from zero to unity is just that of \( u(r, p) \) changing from \( w_0(r) \) to \( w(r) \). This is called deformation, and also \( L(u) \) \( L(w_0) \) and \( A(u) f(r) \) are called homotopic in topology. If, the embedding parameter \( p' \), \( (0 < x < 1) \) is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of eq.(4) can be given as power series in \( p' \) that is

\[ u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \ldots \]  

(7)

The series of eq. (7) is convergent for most of the cases. However, the convergent rate depends on the nonlinear operator \( N(u) \) and the following suggestions have already been made by He \[22\].

(i) The second derivative of \( N(u) \) with respect to \( u \) must be small because the parameter may be relatively large i.e \( p' \to 1 \), and

(ii) The norm of \( L^{-1}\left(\frac{\partial N}{\partial u}\right) \) must be smaller than one so that the series is convergent.

3. Application of HPM: Following \[3\-4,1\], the governing equations of unsteady two dimensional flow of a polytropic gas are given by

\[ \rho u_t + uu_x + vu_y + \frac{p_x}{\rho} = 0 \]  

(8)

\[ \rho v_t + uv_x + vv_y + \frac{p_y}{\rho} = 0 \]  

(9)

\[ \rho \rho_x + v \rho_y + \rho (u_x + v_y) = 0 \]  

(10)

\[ \rho_p + u p_x + v p_y + \gamma (u_x + v_y) = 0 \]  

(11)

with the following initial conditions

\[ u(x,y,0) = e^{x+y}, \quad v(x,y,0) = -1 - e^{x+y}, \]  

\[ \rho(x,y,0) = e^{x+y}, \quad p(x,y,0) = c \]  

(12)

where \( c \) is constant.

To solve equations (8) - (11) by HPM, we construct the following homotopy: 

\[ u = \lim_{p \to 1} u = u_0 + u_1 + u_2 + \ldots \]  

(7)
\begin{align*}
H(u', p') &= (1 - p')[u'_t - u_{ot}] + p' \left[ u'_t + uu'_x + v'u'_y + \frac{p''}{\rho'} \right] = 0 \\
H(v', p') &= (1 - p')[v'_t - v_{ot}] + p' \left[ v'_t + uu'_x + v'u'_y + \frac{p''}{\rho'} \right] = 0 \\
H(\rho', p') &= (1 - p')[\rho'_t - \rho_{ot}] + p' \left[ \rho'_t + uu'_x + v'u'_y + \frac{p''}{\rho'} \right] = 0 \\
H(p'', p') &= (1 - p')[p''_t - p_{ot}] + p' \left[ p''_t + uu'' + v'u'' + \lambda p''(u'_x + v'_y) \right] = 0
\end{align*}

where
\begin{align*}
u' &= u'_0 + p' u'_1 + p'^2 u'_2 + p'^3 u'_3 + \ldotsd \\
v' &= v'_0 + p' v'_1 + p'^2 v'_2 + p'^3 v'_3 + \ldotsd \\
\rho' &= \rho'_0 + p' \rho'_1 + p'^2 \rho'_2 + p'^3 \rho'_3 + \ldotsd \\
p'' &= \rho'_0 + p' p'' + p'^2 p'' + p'^3 p'' + \ldotsd
\end{align*}

The variables \(u, v, \rho, p\) can be obtained similar to eq. (7) as
\begin{align*}
u &= \lim_{p \to 1} u' = u'_0 + u'_1 + u'_2 + u'_3 + \ldotsd \\
v &= \lim_{p \to 1} v' = v'_0 + v'_1 + v'_2 + v'_3 + \ldotsd \\
\rho &= \lim_{p \to 1} \rho' = \rho'_0 + \rho'_1 + \rho'_2 + \rho'_3 + \ldotsd \\
p &= \lim_{p \to 1} p' = p''_0 + p''_1 + p''_2 + p''_3 + \ldotsd
\end{align*}

Substituting the equations (17) to like powers of \(p'\), we get,

Zeroth –order Equation
\[ u'_0 t - u_{ot} = 0 \]

First-order Equation
\[ u'_t + u'_x u'_0 + v'_0 u'_{0y} = 0 \]

Second-order Equations
\[ u'_{2x} + u'_ox u'_{1x} + u'_1u'_{ox} + u'_0v'_{1y} + u'_1v'_{0y} = 0 \]  
(27)

Zeroth –order Equation

\[ v'_{ot} - v'_{0t} = 0 \]  
(28)

First-order Equation

\[ v'_{1t} + u'_ov'_{0x} + v'_0v'_{0y} = 0 \]  
(29)

Second-order Equations

\[ v'_{2t} + u'_ov'_{1x} + u'_1v'_{ox} + v'_0v'_{1y} + v'_1v'_{0y} = 0 \]  
(30)

Zeroth –order Equation

\[ \rho'_{ot} - \rho'_{0t} = 0 \]  
(31)

First-order Equation

\[ \rho'_{1t} + v'_o \rho'_{0x} + v'_0 \rho'_{0y} + \rho'_{1x}u'_ox + \rho'_1u'_oy = 0 \]  
(32)

Second-order Equations

\[ \rho'_{2t} + u'_ov'_{1x} + u'_1v'_{1x} + v'_0 \rho'_{0y} + v'_1 \rho'_{0y} + u'_ox \rho'_{1y} + \rho'_1u'_oy + \rho'_0u'_{1x} + \rho'_0v'_{1y} = 0 \]  
(33)

Solving equations (25) to (33) under the prescribed initial conditions, we get

\[ u'_0 = e^{x+y}, u'_1 = t e^{x+y}, u'_2 = \frac{t^2}{2!} e^{x+y}, \ldots \]  
(34)

\[ v'_0 = -1 - e^{x+y}, v'_1 = \frac{t}{1!} e^{x+y}, v'_2 = -\frac{t^2}{2!} e^{x+y}, v'_3 = -\frac{t^3}{3!} e^{x+y}, \ldots \]  
(35)

\[ \rho'_0 = e^{x+y}, \rho'_1 = \frac{t}{1!} e^{x+y}, \rho'_2 = -\frac{t^2}{2!} e^{x+y}, \rho'_3 = -\frac{t^3}{3!} e^{x+y}, \ldots \]  
(36)

\[ p'_n(x, y, t) = 0, n = 1, 2, 3, \ldots \]  
(37)

The above expressions when \( p' \rightarrow 1 \) are reduced to
\[ u = e^{x+y+t}, \quad v = -1 - e^{x+y+t}, \quad \rho = e^{x+y+t}, \quad p = c \]  

With different values of time \( t \), the exact solutions of \( u(x,y,t) \), \( v(x,y,t) \), \( (x,y,t) \) and \( p(x,y,t) \) are shown through figure 1 to 4, respectively.

**Fig. 1:** Variation of \( u(x, y, t) \) versus \( x, y, t \) when \( c=4 \).

**Fig. 2:** Variation of \( v(x,y,t) \) versus \( x,y,t \) when \( c=4 \).

**Fig. 3:** Variation of \( (x,y,t) \) versus \( x,y,t \) when \( c=4 \).

**Fig. 4:** Variation of \( p(x,y,t) \) versus \( x,y,t \) when \( c=4 \).

**4. Conclusion:** By using HPM scheme, explicit exact solutions arising in nonlinear physics are calculated in the form of a convergent power series having easily computable components. To illustrate the application of this method, numerical results are derived by using the calculated components of the convergent series. The results reported here provide further evidence of the usefulness of Homotopy Perturbation Method (HPM). The HPM is clearly very efficient and powerful technique to find the solutions of the equation governing unsteady flow of a polytropic gas, as the exact solution is reached after few iterations of using HPM. Also this method avoids linearization and biologically unrealistic assumptions, and provides an efficient numerical solution.

**REFERENCES**