Computational Quintic C^4- Lacunary Spline Interpolation Algorithm for Solving Second-order Initial Value Problems

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Abstract: In this paper, a quintic spline interpolation algorithm presented for the solution of second order initial value problems with a new class of the lacunary spline interpolation based on quintic C^4-splines as an approximation to the exact solution of such problems. Convergence analysis of the presented spline function was discussed, stability analysis has been carried out and two examples were considered for the numerical illustration of the presented technique. The results show that the presented quintic spline function which interpolates the lacunary data was efficient and effective for solving such problems.

Key words: Interpolation, Spline function, algorithm, Convergence analysis, initial value problems.

INTRODUCTION

In approximation theory spline functions occupy an important position having a number of applications. Also, spline interpolation is a useful and powerful tool in computer-aided geometric design. Mathematicians have studied many kinds of spline interpolation algorithms to meet the needs of the ever-increasing model complexity and incorporate manufacturing requirements, such as polynomial spline, triangular spline, B-spline, Box spline and others, for example see Chawla and Al-Zanaidi, Khan and Aziz; Jator and Sinkala and Duna, et al. Initial value problems occur in many branches of sciences and engineering, for fluid dynamics, quantum mechanics, optimal problems, etc., for this reason, the numerical solution of this problem is very important because the analytic solution is not always possible. Spline functions of different types has been used by many authors for solving initial values problems, see for example, Sallam and Karaballi; Sallam and Anwar; Kadalbajoo and Patidar; Siddiqi and Akram; Rashidinia and Golbabae; Siddiqi, et. al., and their references.

In this paper, depending on the idea presented by Sallam and Karaballi; Sallam and Anwar; and Saberi, Kordrostami and Esmaeilzadeh, we construct a new algorithm based on a new class of quintic C^4-lacunary spline interpolation and used to find a numerical solution of the second order initial value problems:

\[ y''(x) = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \]

this occurs, for example, in mechanical problems without dissipation and frequently in Celestial Mechanics.

This paper is organized as follows: In section 2, spline function of degree five is presented which interpolates the lacunary data. Some theoretical results about existence, uniqueness and error bounds of the presented spline function are introduced in section 3. Section 4, deals with a new algorithm for calculating quintic C^4- lacunary spline function. Section 5, concerns about the absolute stability of the presented spline. To demonstrate the convergence of the prescribed lacunary spline function, two numerical examples presented in section 6. Finally, in the last section, we prescribed the conclusions of the numerical results.

Construction of the Lacunary Quintic Spline Function: The basic idea in this section is to generate a quintic spline interpolation function \( s(x) \in C^4[0,1] \) which satisfies (1.1) at the knots \( x_i = ih, \quad i = 0, 1, \ldots, N \) and \( h = \frac{1}{N} \). To construct our quintic C^4-spline function, let define:

\[ S_i^q \{ s(x) \} : s \in C^4[0,1], s \in P(x), x \in I = [x, x_n] \]

where \( P(x) \) denotes the set of all polynomials of degree at most 5 and set

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Hereunder, we are going to prove the following theorem about the existence and uniqueness of the new class of quintic $C^r$-lacunary spline function as follows:

**Theorem 2.1:** Given the real numbers, $s_i^\prime, i = 0, 1, \ldots, N$, and $s_i^{\prime\prime}$, then there exist a unique quintic spline $s \in S^4_{n=5}$ such that

\[
\begin{align*}
    s_0 &= f_0, \\
    s_0' &= f_0' \\
    s_i' &= f_i' = f_i'' = 0, i = 0, 1, \ldots, N
\end{align*}
\]

and

\[
    s_i^{\prime\prime} = f_i^{\prime\prime}.
\]

**Proof:** It is easy to verify that the unique quintic spline function $s(x) \in S^4_{n=2}$ defined in $[x_i, x_{i+1}]$ will be written as:

\[
S(x) = A(x)s + hB'(x)s_{i+1} + h^2C(x)
\]

\[
s_i^{\prime\prime} + h^2D(x)s_{i+1}^{\prime\prime} + h^3E(x)s_i^{\prime\prime} + h^3F(x)s_{i+1}^{\prime\prime}
\]

where

\[
A(x) = 1, \quad B(x) = x,
\]

\[
C(x) = \frac{1}{20}(-10x + 10x^2 - 5x^4 + 2x^5),
\]

\[
D(x) = \frac{1}{20}(-10x + 5x^4 - 2x^5)
\]

\[
E(x) = \frac{1}{60}(-5x + 10x^3 - 10x^4 + 3x^5),
\]

\[
F(x) = \frac{1}{60}(5x - 5x^4 + 3x^5),
\]

and $x = x_i + th$, $0 \leq t \leq 1$ with a similar expression for $s(x)$ in $[x_{i-1}, x_i]$. Since $s \in C^4[0, 1]$, and $S'(x_i^+) = S'(x_i^-) \to S^{(4)}(x_i^+) = S^{(4)}(x_i^-)$ respectively, for $i = 0, 1, 2, \ldots, N$, leads to the following linear system of equations:

\[
s_i = s_{i-1} + hs_i^{\prime} - \frac{h^2}{20} \left[ 3s_{i-1}^{\prime\prime} + 7s_i^{\prime\prime} \right] + \frac{h^3}{60} \left[ -2s_i^{\prime\prime\prime} + 3s_i^{\prime\prime} \right],
\]

\[
s_{i+1} = s_{i+1}^{\prime} + \frac{h}{2} \left[ s_i^{\prime\prime} + s_{i+1}^{\prime\prime} \right] + \frac{h^2}{12} \left[ s_i^{\prime\prime\prime} - s_{i+1}^{\prime\prime\prime} \right],
\]

\[
s_i^{\prime\prime} = f(x_i + s_i), \quad s_i^{\prime\prime} = s_{i-1}^{\prime\prime},
\]

and

\[
2s_i^{\prime\prime} + 8s_i^{\prime\prime} + 2s_i^{\prime\prime} = \frac{6}{h^2} [s_i^{\prime\prime} - s_i^{\prime\prime} ], \quad i = 1, 2, \ldots, N,
\]

and hence $S(x)$ is uniquely determined in $[0, 1]$. Based on the above scheme (2.4), we are generate a quintic spline function $s(x) \in C^4[0, 1]$, to find an approximation to the exact solution of the second order initial value problem (1.1). Let $f \in C^4([0, 1] \times R)$ and satisfy the Lipschitz condition

\[
|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|
\]

or $|e_{i+1}^{(q)}| \leq L|e_i^{(q)}|, \quad q = 0, 1, 2, 3$ (2.5)

for $x \in [0, 1]$, and all real numbers $y_1, y_2$.

Now from (2.1)-(2.4), the approximate quintic spline solution $s(x)$ to the exact solution of (1.1) will be constructed for $i = 1, 2, \ldots, N$ as follows:

\[
s_i = s_{i-1} + hs_i^{\prime} - \frac{h^2}{20} \left[ 3f(x_{i-1}, s_{i-1}) + 7f(x_i, s_i) \right] + \frac{h^3}{60} s_i^{\prime\prime},
\]

\[
s_i^{\prime} = s_i^{\prime} + \frac{h}{2} \left[ f(x_{i-1}, s_{i-1}) + f(x_i, s_i) \right],
\]

(2.6a)

(2.6b)
\[ s_i' = f(x_i + s_i), \quad (2.6c) \]

and

\[ s_{i+1}'' = -5s_i'' + \frac{3}{h} \left[ f(x_{i+1}, s_{i+1}) - f(x_{i-1}, s_{i-1}) \right], \quad (2.6d) \]

where \( s_i^{(q)} \) for \( q = 0, 1, 2, 3 \) and \( i = 1, 2, \ldots, N \) can be found by solving the above system \((2.6a)-(2.6d)\) using the contraction mapping principle. Using the contraction mapping principle, it can be easily verified that the spline function approximation \( s(x) \), is successfully uniquely determined using the above recurrence formula for all \( h \) satisfying \( Lh(h + 6) < 1 \).

**Convergence Analysis:** Our purpose in this section is to investigate the convergence analysis of quintic \( C^4 \) spline interpolation \( s(x) \) used to solve the second order initial value problems, and the emphasis conditions for convergence will be established. Also the upper bounds for errors \( e(x_i) = s(x_i) - f(x_i) \) where \( i = 1, 2, \ldots, N \) and its derivatives are presented.

The following lemmas and theorem shows the upper bounds for errors of the presented quintic \( C^4 \) spline function in section 2 where \( W_r(h) \) for \( r = 3, 5 \) denotes the modulus of continuity of \( f(x) \) in \([0, 1]\);

**Lemma 3.1:** If \( Lh(h + 6) < 1 \) and \( f(x) \in C^5[0,1] \) then we have

\[ |e_i| \leq \frac{h^5}{60} \left[ \frac{-180 - 35L - 28Lh^2 + 9Lh^3}{1200 + 164Lh^2 - 220Lh^3 - 63Lh^4 + 47Lh^5 + 8Lh^6} \right] W_5(h), \]

and

\[ |e_i'| \leq \frac{h^4}{12} \left[ \frac{60 + 21L + 24Lh^2 + Lh^3}{-60 + 9Lh^2 + 8Lh^3} \right] W_5(h). \]

**Proof:** We will consider \( f(x) \in C^5[0,1] \), and

\[ e_0^{(r)} = s^{(r)}(x_0) - f^{(r)}(x_0) = 0, \quad r = 0, 1, 2, 3. \]

Using \((2.5)\), the system \((2.4)\), for \( i = 1 \) becomes:

\[ s_1 = s_0 + hs_1' - \frac{h^2}{20} \left[ 3s_0'' + 7s_1'' \right] + \frac{h^3}{60} \left[ -2s_0''' + 3s_1''' \right], \]

and

\[ s_1' = s_0' + \frac{h}{2} [s_0'' + s_1''] + \frac{h^2}{12} [s_0''' - s_1''']. \]

The

\[ s_1 = s_0 + hs_1' - \frac{h^2}{20} \left[ 3s_0'' + 7s_1'' \right] - \frac{h^2}{20} \left[ 3s_0''' + 7s_1''' \right] + \frac{h^3}{60} \left[ -2s_0'''' + 3s_1'''' \right] + \frac{h^4}{120} \left[ -2s_0'''''' + 3s_1'''''' \right]. \]

From equation \((3.1)\) and \((3.2)\) and by applying Taylor’s expansion formula, and employing \((2.5)\) with lemma 2.1 in Sallam and Karaballi\(^{1(2)}\), yields:

\[ |e_1| \leq \frac{7}{20} h^2 L + \frac{1}{20} h^3 L \left| e_1' \right| \leq \frac{h^3}{15} W_5(h), \]

and

\[ \left| e_1' - \frac{hL}{2} e_1 - \frac{1}{12} h^2 L e_1 \right| \leq \frac{h^4}{12} W_5(h). \]

Now solving these two equations \((3.3)\) and \((3.4)\) simultaneously, yields:
Hence the lemma 3.1 is proved.

Lemma 3.2: Let \( f(x) \in C^4[0,1] \) then 
\[
|e_i^{(r)}| \leq O(h^{3-r})
\]
where \( e_i^{(r)} = S_i^{(r)} - f_i^{(r)} \)

Proof: From the first equation of the system (2.4) and by applying Taylor’s series formula, we obtain
\[
e_i = e_{i-1} + h e_i - \frac{h^2}{20} \left[ 3e_i^" + 7e_{i-1}^" \right] + \frac{h^3}{60} \left[ 3e_i^" - 2e_{i-1}^" \right] + \frac{h^5}{15} (f^{(5)}(\alpha_1) - f^{(5)}(\alpha_2)),
\]
where \( \alpha_1, \alpha_2 \in [x_i, x_{i+1}] \).

Hence,
\[
|e_i| \leq O(h^5) \tag{3.6}
\]
Similarly, using the second and third equations of the system (2.4) by and applying Taylor’s expansion formula, we obtain
\[
|e_i^{(r)}| \leq O(h^4),
\]
and
\[
|e_i^{(r)}| \leq O(h^2).
\]

Theorem 3.1: Let \( f(x) \in C^5[0,1] \) then for all \( x \in [0,1] \), we have
\[
|S^{(r)} - f^{(r)}| \leq C_i h^{6-r}, \quad \text{where } r = 1,2,3,4,5 \text{ and } r = 0,1,2,3,4.
\]

where \( C_i \) denote generic constants independent of \( h \), but dependent on the order of the various derivatives.

Proof: On \( X \in [x_i-1, x_i] \) with mesh size \( h \), and if 
\[
S(x) \in C^4[0,1],
\]
we have
\[
e^{(4)}(x) = S^{(4)}(x) - y^{(4)}(x) = S^{(4)}(x) - u^{(4)}(x) + y^{(4)}(x),
\]
where \( u^{(4)}(x) \) is the linear interpolant of \( y^{(4)}(x) \) at \( x_{i-1} \) and \( x_i \). see Khan and Aziz\(^{7}\) for the fourth class, can be written as
\[
u^{(4)}(x) = \frac{1}{h^2} \left[ C^{(4)}(x)y_{i-1}^{''} + \frac{1}{h^2} D^{(4)}(x)y_i^{''} + \frac{1}{h^2} E^{(4)}(x)y_{i+1}^{''} + \frac{1}{h^2} F^{(4)}(x)y_i^{''} \right],
\]
and from (2.3) we have
\[
S^{(4)}(x) = h^2 (-6 + 12x) s_i^{''} + h^3 (6 - 12x) s_i^{'''} + h^4 (-4 + 6x) s_i^{(4)},
\]
\[
h^2 (-6 + 12x) e_i^{''} + h^3 (-4 + 6x) e_i^{'''} + h^4 (-2 + 6x) e_i^{(4)} \tag{3.7}
\]
From (3.7) and (3.8), we obtain
\[
h^2 \left| S^{(4)}(x) - u^{(4)}(x) \right| \leq h^2 \left| C^{(4)}(x) \right|
\]
\[
+ \left| e_i^{''} \right| + \left| e_i^{'''} \right| + h^2 \left| D^{(4)}(x) \right| \left| e_i^{''} \right| + h^3 \left| F^{(4)}(x) \right| \left| e_i^{'''} \right| + h^4 \left| F^{(4)}(x) \right| \left| e_i^{(4)} \right|
\]
\[
\leq 6h^2 \left| e_i^{''} \right| + 6h^2 \left| e_i^{'''} \right| + 4h^2 \left| e_i^{(4)} \right| + 4h^3 \left| e_i^{(5)} \right| + 4h^4 \left| e_i^{(6)} \right| \tag{3.9}
\]
Now, using lemma 3.2, and applying (3.9) we obtain
\[
|S^{(4)}(x) - u^{(4)}(x)| = O(h).
\]

Using Taylor’s expansion formula, it is easy to verify that
\[
|y^{(4)}(x) - y^{(4)}(x)| \leq h^2 \left| W_S(h) + 2 \left| y^{(5)}(x) \right| \right|,
\]
and hence
\[ |S^{(4)}(x) - y^{(4)}(x)| \leq |S^{(4)}(x) - u^{(4)}(x)| + C_j h \]
\[ |u^{(4)}(x) - y^{(4)}(x)| \leq C_j h \]  \hspace{1cm} (3.10)

Similarly, we have
\[ h^3 |S^{(3)}(x) - u^{(3)}(x)| \leq \frac{3}{2} h^2 |e_i^3| + \frac{3}{2} h^3 |e_i^2| + h^3 |e_i^1| \]

Hence
\[ |S^{(3)}(x) - y^{(3)}(x)| \leq C_j h^2 \]  \hspace{1cm} (3.11)

For any \( x \in [x_{i-1}, x_i] \), using (3.11), we have
\[ S^{(2)}(x) - y^{(2)}(x) = \int_{x_{i-1}}^{x} [S^{(3)}(x) - y^{(3)}(x)] \, dx + e_i^3 \]

Using lemma 3.2 yields:
\[ |S^{(2)}(x) - y^{(2)}(x)| \leq C_j h^3 \]

Integrating once more over \([x_{i-1}, x_i]\), and using lemma 3.2, yields:
\[ |S^{(1)}(x) - y^{(1)}(x)| \leq C_j h^4 \]

and
\[ |S^{(0)}(x) - y(x)| \leq C_j h^5 \]

**Algorithm C^{i0} - quintic Spline:**

**Step 1:** Partition \([0, 1]\) to \(N\) subintervals \(I_i\).

**Step 2:** Set \(s(0) = y_0 = y^{(0)}(0), s'(0) = y^{(1)}(0), s''(0) = f(0, y_0)\)

and evaluate \(S''_0 = y''(0)\) of \(f(x, y')\) at \(x_0\).

**Step 3:** For \(i = 1, 2, \ldots, nN\) do \(S''_i = f(x_i, S'_i)\).

**Step 4:** Let \(x \in [0, 1]\) and \(i=1,\ldots, N\).

**Step 5:** If \(x \in [x_{i-1}, x_i]\) go to step 6, else \(i=i+1\) and repeat this till find a proper \(i\).

**Step 6:** Set \(k=i\).

**Step 7:** Set \(i=1\) computing \(S_i, S'_i\) at \(N+1\) equally spaced points in each subinterval \([x_i, x_{i+1}]\), \(i=1, 2, \ldots, N\).

\[ s_i = s_{i-1} + hS'_i - \frac{h^2}{20} \left[ 3f(x_{i-1}, s_{i-1}) + 7f(x_i, s_i) \right] + \frac{h^3}{60} s''_i \]

and
\[ s''_i = s''_{i-1} + \frac{h}{3} \left[ f(x_{i+1}, s'_i) - f(x_{i-1}, s'_i) \right] \]

**Step 8:** Compute
\[ s''_{i+1} = -5s''_i + \frac{3}{h} \left[ f(x_{i+1}, s'_{i+1}) - f(x_{i-1}, s'_{i-1}) \right] \]

**Step 9:** Set \(i=i+1\)

**Step 10:** If \(i=n+1\) go to step 11, else go to step 7.

**Step 11:** Stop.

**Note:** Step 7 for solving our system, the initial values of \(S'_i, S''_i\) are needed, which can be evaluated from Taylor series expansion as

\[ s_i = s_{i-1} + hs'_{i-1} + \frac{h^2}{2} s''_{i-1} + \frac{h^3}{6} s'''_{i-1} + \frac{h^4}{24} s^{(4)}_{i-1} \]

and

\[ s''_{i+1} = s''_{i-1} + \frac{h^2}{2} s'''_{i-1} + \frac{h^3}{6} s^{(4)}_{i-1} + \frac{h^4}{24} s^{(5)}_{i-1} \]

**Absolute Stability:** In this section, the stability analysis of the presented method (2.6a)-(2.6d) will be considered, and applying the method to the test equation

\[ y'' = -\lambda^2 y, \]
\( \lambda \in \mathbb{R}, \ y(x_0) = y_0, \ y'(x_0) = y'_0 \)

(5.1)

Setting \( \lambda h = H \) using (5.1) to obtain

\[
s_i = s_{i-1} + \frac{h^2}{20} \left[ 3\lambda^2 s''_{i+1} + 7\lambda^2 s''_{i} \right] + \frac{h^3}{60} \left[ s'''_{i} \right]
\]

and

\[
h s'_i = h s'_i - \frac{h^2}{2} \left[ \lambda^2 s_{i-1} + \lambda^2 s_i \right]
\]

and

\[
h^3 s''_i = \frac{3}{5} h^2 \lambda^2 [s_{i-1} - (1 + h)s_i] - \frac{h^3}{5} s'''_{i-1}
\]

Or in matrix notation

\[
S = AS, \quad i=1,2,\ldots, N,
\]

where

\[
S = \begin{pmatrix}
S_i \\
h s'_i \\
h^3 s''_i
\end{pmatrix}, \quad S_i = \begin{pmatrix}
S_{i-1} \\
h s'_{i-1} \\
h^3 s''_{i-1}
\end{pmatrix}
\]

Then the characteristic equation can be written as:

\[
r^3 - (tr A) r^2 + (A_1 + A_2 + A_3) r - \det(A) = 0
\]

(5.2)

where \( r \) is the eigenvalue, and \( A_1, A_2, A_3 \) denotes respectively the cofactors of the diagonal elements. The quintic spline approximation method defined by (2.6a)-(2.6b) have interval of periodicity \((0, H^2)\) where the eigenvalues \( r_{1,2} \) of the matrix \( A \) are complex conjugate and \( |r|<1 \). The characteristic polynomial (5.2) is said to be stable if all complex eigenvalues have negative real parts.

**Numerical Illustrations and Discussions:** To illustrate our quintic \( C^1 \)-lacunary spline function and to demonstrate the applicability of our presented method computationally, we considered two initial value problems of second order whose exact solutions are known to us. The applicability of the results shows that the effectiveness of the proposed technique. The notation \( e', e'' \) and \( e''' \) stands for the maximum magnitude errors

\[
|s'(x) - y'(x)|, \ |s''(x) - y''(x)| \text{ and } |s'''(x) - y'''(x)|
\]

respectively.
**Example 6.1:** Mathews and Fink\[9\]
Consider the problem
\[
\begin{align*}
  y'' + 25y &= 8\sin(x) \\
  y(0) &= 0 = y'(0) \quad \text{where } x \in [0, 1]
\end{align*}
\]
The exact solution is
\[
y(x) = \frac{4}{25}\sin(5x) - \frac{4}{5}\cos(5x).
\]

**Example 6.2:** (Sallam and Anwar (2000))
Consider the problem
\[
\begin{align*}
  y'' &= x + y \\
  y(0) &= 1, y'(0) = 0 \quad \text{where } x \in [0, 1]
\end{align*}
\]
The exact solution is \( y(x) = e^x - x \).

**Table 1:** Absolute maximum error for \( s(x) \) and it’s derivatives with different values of \( h \) for Example 6.1

<table>
<thead>
<tr>
<th>( h )</th>
<th>( e )</th>
<th>( e' )</th>
<th>( e'' )</th>
<th>( e''' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.25\times 10^{-4}</td>
<td>8.1\times 10^{-5}</td>
<td>1\times 10^{-4}</td>
<td>1.4\times 10^{-6}</td>
</tr>
<tr>
<td>0.05</td>
<td>1\times 10^{-5}</td>
<td>5\times 10^{-6}</td>
<td>2.9\times 10^{-7}</td>
<td>4.6\times 10^{-9}</td>
</tr>
<tr>
<td>0.01</td>
<td>3.3\times 10^{-9}</td>
<td>8.3\times 10^{-10}</td>
<td>1\times 10^{-10}</td>
<td>4.4\times 10^{-12}</td>
</tr>
<tr>
<td>0.001</td>
<td>1\times 10^{-13}</td>
<td>8.3\times 10^{-14}</td>
<td>6.67\times 10^{-16}</td>
<td>7.9\times 10^{-18}</td>
</tr>
</tbody>
</table>

**Table 2:** Absolute maximum errors for \( s(x) \) and it’s derivatives with different values of \( h \) for Example 6.2

<table>
<thead>
<tr>
<th>( h )</th>
<th>( e )</th>
<th>( e' )</th>
<th>( e'' )</th>
<th>( e''' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.4\times 10^{-6}</td>
<td>8.4\times 10^{-7}</td>
<td>1\times 10^{-9}</td>
<td>1.2\times 10^{-11}</td>
</tr>
<tr>
<td>0.05</td>
<td>2\times 10^{-7}</td>
<td>1\times 10^{-8}</td>
<td>3.7\times 10^{-10}</td>
<td>2.7\times 10^{-12}</td>
</tr>
<tr>
<td>0.01</td>
<td>3.2\times 10^{-10}</td>
<td>9\times 10^{-11}</td>
<td>8.4\times 10^{-13}</td>
<td>1\times 10^{-15}</td>
</tr>
<tr>
<td>0.001</td>
<td>2.9\times 10^{-11}</td>
<td>4.5\times 10^{-12}</td>
<td>1.66\times 10^{-14}</td>
<td>2.25\times 10^{-16}</td>
</tr>
</tbody>
</table>

**Conclusion:** The scarcity of studies on the application of the lacunary spline function, led to propose the quintic \( C^1 \)-lacunary spline function for finding the numerical solution of the second order initial value problems. As it can be seen in the above two examples, our presented algorithm provides encouraging results and also yields a good approximation to the solution provided that, a small step size \( h \) must be chosen. The new algorithm presented in this paper is not only approximates the solution of the second order initial value problems, but it also approximates the higher order derivatives as well. The possibility of finding the solution of further higher order initial value problems can also be explored in future.

**REFERENCES**


