Magnetic Field Mapping of a Direct Current Electrical Machine Using Finite Element Method

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Abstract: Magnetic flux pattern and magnetization curve of dc machine are obtained using an advance numerical method called Finite Element Method. The method is much suited for dealing with the complicated internal structure of the machine and non linear magnetic characteristics. Finite Element Method Magnetic software Version 4.0 (FEMM) is applied to two dimensional structure of the machine. The result of the computation are plotted to produce intricate flux patterns in the machine domain. The open circuit characteristics obtained agree with the conventional curve.

Key words: Finite Element Method Magnetic, Direct Current Machine, Partial Differential Equation with boundary conditions, Poisson Equations.

INTRODUCTION

Analysis and prediction of dc machine magnetic field pattern and characteristics are difficult due to irregular geometry and non-linear magnetic materials associated with this machine. The d. c machine has its field poles on the stator with armature and commutator on the rotor. Localized flux density variations within the pole shoe produces hysteresis and eddy-current losses as the teeth of the rotor sweep across the pole faces[1]. In order to obtain accurately the performance of this machine at its design stage, the magnetic field mapping of the internal structure and performance characteristics are necessary. Different methods have been deployed over the years to handle this task. These are the Finite Difference Method (FDM), Boundary Element Method (BEM) and the Finite Element Method (FEM). However the FEM is increasingly being used by designers due to its ability to handle the complex internal structure and non-linear materials present in the machine structure[5].

In practice, magnetic saturation has made the characteristics of these machines to be non-linear as such require accurate method of modeling and computation. Consequently the machines calculations are based on approximation that typified the operating point. Inaccuracies resulting from these approximations make it necessary to employ high precision analytical tool – the finite element method.

The proposed methodology is to compute the flux pattern in a four pole, direct current machine and obtain magnetization curve and magnetic flux density in the air gap.

Statement of Problem: The basic problem of the electromagnetic field computation that resulted in the magnetic field mapping/distribution in the d. c machine cross section machine is modeled by the two dimensional Poisson Equation as follows:

\[
\frac{\partial}{\partial x} \left[ \frac{1}{\mu \mu_0} \frac{\partial A}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \frac{1}{\mu \mu_0} \frac{\partial A}{\partial y} \right] = -J
\]

\[
\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -\mu J
\]  (1)

Where:

\(\mu_0\) = Permeability of free space
\(\mu_r\) = Relative Permeability
\(\mu\) = Absolute permeability = \(\mu_0\mu_r\)

A\(_n\) = Magnetic vector potential normal to the section of the machine

J\(_n\) = current density vector normal to the section

However, vector magnetic potential is of great value when solving two-dimensional problems containing current carrying areas[6]. This equation is reformulated by variational calculus, in the finite element method, and first-order triangular elements are used to discretize the field region, resulting in a set of linear algebraic equations[6]. These linear simultaneous equations are solved using Newton-Raphson technique to obtain the magnetic vector potential.
Mathematical Formulation: Finite Element Method as applied to this work is used to solve derived electromagnetic field problems of Poisson’s type from basic magnetostatic Maxwell equations. The equations relate magnetic vector potential A, magnetic flux density B and magnetic field intensity H to obtain the Poisson’s equation.

3.1 Maxwell’s Equation:
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad (2) \]
\[ \nabla \cdot \mathbf{B} = 0 \quad (3) \]
And the relationship
\[ \mathbf{B} = \mu \mathbf{H} \]
Where B is the magnetic flux density, H is the magnetic field intensity, J is the current density and \( \mu \) is the material absolute permeability.

If a vector field has no divergence, that vector field is the curl of some other vector fields\(^3\). Therefore, \( \nabla \cdot \mathbf{B} = 0 \) means that B is the curl of another vector field. Let that other vector field be A. Since \( \nabla \cdot \mathbf{B} = 0 \), there exists a magnetic vector A such that \( \mathbf{B} = \nabla \times \mathbf{A} \).

A current in the z direction produces A\(_z\) only, and x and y component of B, A is then related to the flux circulating in the x, y plane, per unit length in the z direction; it has the units of Webers per metres length. Thus for two-dimensional field, A can be treated as a scalar quantity. The magnetic flux flowing in a conducting iron core carrying current has its permeability \( \mu \) greater than that of copper conductor of permeability \( \mu_c \).\(^3\) Solving for the magnetic vector potential A, the component of the magnetic flux density are obtained by finding the derivatives
\[ \frac{\partial \mathbf{A}}{\partial x} \quad \text{and} \quad \frac{\partial \mathbf{A}}{\partial y} \]
as shown below.
\[ \mathbf{B} = \nabla \times \mathbf{A} \quad (4) \]
\[ \mathbf{B} = \mu \frac{\partial \mathbf{A}_x}{\partial y} - \mu \frac{\partial \mathbf{A}_y}{\partial x} \quad (5) \]

With current density J\(_z\) only, and hence A\(_z\) only, and no variation of these quantities in the z direction,
\[ \mathbf{B} = \frac{\partial \mathbf{A}_x}{\partial y}, \quad \mathbf{B}_z = - \frac{\partial \mathbf{A}_y}{\partial x} \quad (6) \]

\[ \nabla \times \mathbf{H} = \mathbf{J} \]
\[ \mathbf{J} = \mathbf{0} \quad \text{and} \quad \mathbf{J}_x = 0, \quad \text{do not exist} \]
Therefore only \( \mathbf{J} = \frac{\partial \mathbf{H}}{\partial x} - \frac{\partial \mathbf{H}}{\partial y} \) exist

Using \( \mathbf{B} = \mu \mathbf{H} \) and substituting equation 6 into the above equation gives
\[ \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial \mathbf{A}}{\partial x} \right) + \frac{\partial}{\partial y} \left[ \frac{1}{\mu} \frac{\partial \mathbf{A}}{\partial y} \right] = -\mathbf{J} \]
And so for fixed \( \mu \)
\[ \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} = -\mu \mathbf{J} \quad (8) \]
which is Poisson’s Equation

3.2 Shape Function and Discretization: Finite element method employs discretization of the solution domain into smaller regions called elements, and the solution is determined in terms of discrete values of some primary field variables A (example vector magnetic potential in x, y, z directions) at the nodes. The number of unknown primary field variables at a node is the degree of freedom at that node. The discretized domain comprised of triangular shaped element as shown in figure 1(a) with each node having one degree of freedom as shown in the figure 1(b). The two-dimensional Cartesian coordinate system presented in this paper uses triangular element with nodes only at corners as shown in figure 1(a). Magnetic potentials A\(_x\), A\(_y\), and A\(_z\) are assumed to exist at the nodes. It is now necessary to define the variation of this magnetic potential over the element, and this is known as Shape Function\(^6\).

The shape function is represented by a polynomial approximation consistent with the number of nodes (and associated potentials) on each edge. The polynomial approximation for magnetic vector potential (A) used to model a four pole dc machine is expressed by the general term \( a_{i} \), \( a_{j} \), and \( a_{k} \) are coefficients dependent on A\(_x\), A\(_y\) and A\(_z\) and their associated coordinate. These equations are available to determine the three unknown \( a_{i} \), \( a_{j} \), and \( a_{k} \).
Fig. 1(a): Discretization of the field region by triangular finite element

Fig. 1(b): A domain of One Element

\[ A_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_j \]
\[ A_j = \alpha_1 + \alpha_3 x_j + \alpha_2 y_j \]
\[ A_k = \alpha_1 + \alpha_2 x_1 + \alpha_1 y_1 \]  

from which

\[ \alpha_1 = \frac{(aA + aA + aA)}{2\Delta} \]  

where

\[ \Delta = \text{area of element} \]

\[ = \frac{(a + b x_i + c y_j)}{2} \]
Fig. 2: Non linear B-H curve and its energy

\[ \alpha_2 = \frac{(bA + bA + A)}{2\Delta} \]  \hspace{1cm} (14)

where

\[ b_i = (y_i - y_1) \]
\[ b_j = (y_j - y_1) \]
\[ b_k = (y_k - y_1) \]  \hspace{1cm} (15)

and

\[ \alpha_3 = \frac{(cA + cA + cA)}{2\Delta} \]  \hspace{1cm} (16)

\[ c_i = (x_i - x_j) \]
\[ c_j = (x_j - x_k) \]
\[ c_k = (x_k - x_i) \]  \hspace{1cm} (17)

Substituting for \( \alpha_i, \alpha_i, \) and \( \alpha_i \) in equation 9

\[ A = \frac{[(a + bx + cy)A + (abx + cy)A + (a + bx + cy)A]}{2\Delta} \]
\[ = \sum_{m=i,j,k} \frac{1}{2\Delta} (a + bx + cy)A \]  \hspace{1cm} (18)
\[ = \sum_{m=i,j,k} \frac{1}{2\Delta} N A \]

where summation \( m = i, j, k \) indicates the summation as \( m \) takes the values \( i, j, k \) in turn and \( N \) etc are the Shape Functions.

3.3 Variational Method: The variational method is essentially an energy method; it is useful to consider what is meant by the term energy and its associated co energy in relation to magnetic devices.

In general, magnetic materials are non linear and the magnetic energy needed to generate a flux density \( B \) in the material, corresponds to the figure below. The term energy will imply the energy stored in the magnetic field and this corresponds to the area \( W_e \).

It will be equal to

\[ \int_{0}^{h} db \]  \hspace{1cm} (19)

The area \( C_e \) denotes the co-energy, which corresponds to

\[ \int_{0}^{h} d \]  \hspace{1cm} (20)

The magnetic store energy is equal to the net energy supplied by the source. The total area \( C_e + W_e \) will be equal to the integration over the exciting coil of \( JA \)

Where \( J \) is the final current density and \( A \) final flux levels

Therefore co-energy density may be written locally as

\[ C_e = JA - \int_{0}^{h} db \]  \hspace{1cm} (21)

In formulating problems in terms of vector magnetic potential \( A \), coil currents are usually known.
and the appropriate energy functional to be minimized is the co-energy quantity

\[ C' = \int \left[ \mathbf{J} \mathbf{A} - \frac{\nabla}{\nabla} \mathbf{f} \mathbf{b} \mathbf{d} \mathbf{b} \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \mathbf{d} \mathbf{z} \right] \]  

(22)

Since reluctivity \( \nu = \frac{\mu}{b} \), equation 22 can be written as

\[ C' = \int \left[ \mathbf{J} \mathbf{A} - \frac{\nabla}{\nabla} \mathbf{f} \mathbf{b} \mathbf{d} \mathbf{b} \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \mathbf{d} \mathbf{z} \right] \]  

(23)

Two-dimensional case is considered in two dimensional structure. The term containing flux density is expressed in terms of the vector magnetic potential \( \mathbf{A} \). As before it will be assumed that the reluctivity at any point can be assigned a value related to the final flux density \( \mathbf{B} \). Thus the functional becomes

\[ C' = \int \left[ \mathbf{J} \mathbf{A} - \frac{\nabla}{\nabla} (\mathbf{B}^*) \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

(24)

Comparing equation 24 with a parabola

\[ f(x) = \frac{1}{2} x^2 - x^2 \]  

whose minimum is at \( x = f \)

Considering one element only as before

\[ F_e = \int \left[ \mathbf{J} \mathbf{A} - \frac{\nabla}{\nabla} \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 + \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

(25)

Hence, the contribution to the rate of change of \( F \) with \( A \) from the variation of potential of node \( i \) in element \( e \) only

\[ \frac{\partial F_e}{\partial A} = \int \frac{\partial}{\partial A} \left[ \mathbf{J} \mathbf{A} - \frac{\nabla}{\nabla} \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 + \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

\[ = \int \left[ \mathbf{J} \frac{\partial \mathbf{A}}{\partial A} - \frac{\nabla}{\nabla} \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 + \left( \frac{\partial \mathbf{A}}{\partial x} \right)^2 \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

(26)

Substitute shape function equation derived from figure 1(b) and equation 9

\[ \mathbf{A} = \sum_{i=1}^{m} \left[ \begin{array}{c} (a_i + b x + c y) A_i \\ \frac{\partial A_i}{\partial x} \\ \frac{\partial A_i}{\partial y} \end{array} \right] \frac{1}{2 \Delta} \]  

(27)

Substitute equation 27 into equation 26

\[ \frac{\partial F_e}{\partial A_{in}} = \frac{1}{2 \Delta} \left[ \frac{\partial}{\partial A_{in}} \left( \sum_{i=1}^{m} \left[ \begin{array}{c} (a_i + b x + c y) A_i \\ \frac{\partial A_i}{\partial x} \\ \frac{\partial A_i}{\partial y} \end{array} \right] \right) \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

or

\[ \frac{\partial F_e}{\partial A_{in}} = \frac{1}{2 \Delta} \int \left[ \frac{\partial}{\partial A_{in}} \left( \sum_{i=1}^{m} \left( a_i + b x + c y \right) A_i \right) \right] \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} \]  

(29)

The equation 29 can further be expressed with respect to each of the nodal potential. The integral equation 29 can be readily evaluated for the two-dimensional basic functions defined by equation 27 by using the expression of equation 30.

\[ \int \mathbf{M}_i \mathbf{N}_j \mathbf{N}_k \mathbf{A}_m \mathbf{d} \mathbf{x} \mathbf{d} \mathbf{y} = \frac{1}{(\alpha + \beta + \gamma)!} 2 \Delta \]  

(30)

With \( J \) which is constant within the element, it can be shown that

\[ \frac{1}{2 \Delta} \int \left[ (a_i + b x + c y) \right] = \frac{J \Delta}{3} \]  

The integral over the area of the element can be shown to be

\[ \frac{\partial F_e}{\partial A_{in}} = \frac{J \Delta}{3} \left[ \begin{array}{c} b b + c c + b c + c b + c c \cdot A_i \\ + c b + c c + b c + c b + c c \cdot A_i \right] \]  

(31)
The minimization of the equation 29 is carried out for all the triangles of the field region on the cross section of the generator, the matrix equation 32 is obtained, the solution determines the unknown vector potential \( \mathbf{A} \).

\[
\frac{\partial F}{\partial \mathbf{A}_n} \bigg|_{n,j,k} = [T_i][S_i][A_i]
\]  

(32)

where \([T_i] = \frac{J\Delta}{3}\), the source term with constant \( J \) within the element

\[
[S_i] = \frac{V}{4\Delta^3} \begin{bmatrix}
S_{a}\ S_{b}\ S_{c}
S_{b}\ S_{a}\ S_{c}
S_{c}\ S_{a}\ S_{b}
\end{bmatrix}
\]  

(33)

\( S_a = b^2 + c^2, S_b = b^2 + c^2, S_c = b^2 + c^2, \)

\( S_a = b^2 + c^2, S_b = b^2 + c^2, S_c = b^2 + c^2, \)

\( S_a = b^2 + c^2, S_b = b^2 + c^2, S_c = b^2 + c^2, \)

\( S_a = b^2 + c^2, S_b = b^2 + c^2, S_c = b^2 + c^2, \)

\( S = \text{Non-linear matrix. The non-linear equation 32 is first quasi-linearized by a Newton-Rapson method, and the resulting in a set of equations solving directly or iteratively using digital computers.} \)

\[
[A_i] = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}
\]  

(34)

Equation 34 is known as magnetic vector potential at the three nodes of the triangular element

\[ A \]  

3.4 Boundary Conditions: The numerical equations considered above have to be solved subject to the relevant boundary conditions. The types of boundary conditions considered in this paper are summarized below:

**Dirichlet:** A Dirichlet boundary condition set the unknown function to a known on the boundary of the partial differential equation. This is the boundary condition that the value of the potential \( A \) is explicitly defined on the boundary, example \( A = 0 \). The most common use of Dirichlet-type boundary conditions in magnetic problem is to define \( A = 0 \) along a boundary to keep magnetic flux from crossing the boundary\(^5\).

**Neumann:** Neumann boundary condition specifies the normal derivative of the potential along the boundary. In magnetic problem, the homogeneous Neumann boundary condition, \( \frac{\partial A}{\partial n} = 0 \) is defined along a boundary to force flux to pass the boundary at exactly 90° angle to the boundary. This sort of boundary condition is consistent with an interface with a very highly permeable metal\(^5\).

3.5 Solution of the Problem: There are three unknown vector potential at each node of the triangular elements. The complete solution of the field problem consists of the solution of the system in equation 30 for each finite element that belongs to the domain. It is necessary to pass from the local matrix [S] that refers to the single \( m^{th} \) element (i.e. from the three value \( A_m, A_n, \) and \( A_l \) only) to the global matrix that refers to the complete domain (i.e., to all the value \( A_m \) of the \( N_m \) nodes).

Since the potential is vector magnetic potential \( A \), the flux density \( B \) depends on the potential gradient and the flux density remain constant within the element.

The governing differential equation is now applied to the domain of a single element. At the element level, the solution to the governing equation is replaced by continuous function approximating the distribution of \( A \) over the element domain, expressed in terms of the unknown nodal value \( A_1, A_2, A_3 \), of the solution \( A \). A system of equations in terms of \( A_1, A_2, \) and \( A_3 \) will then be formulated for the element. Once the element equations are determined, the elements are assembled to form the entire domain of the problem. The solution \( A(x,y) \) to the problem becomes a polynomial approximation, expressed in terms of the nodal values of \( A \). A system of linear algebraic equations results, which may be solved by means of common numerical algorithms.
DC Machine Field Computation and Solution: Finite Element Method Magnetics (FEMM) is a suite of programs for solving low frequency electromagnetic problem on two-dimensional planar and axi-symmetric domain[5].

The procedure for implementing this numerical computation of magnetic field problems is by using the finite element method which is divided into three main steps:

Pre-processing: This processor is used for drawing the problems geometry, defining materials and defining boundary conditions.

The derivation of the finite element model of this machine under consideration involves defining conduction materials, electromagnetic materials and their properties and boundary conditions and eventually resulted in mesh generation. The machine is made up of two sections, the rotor and stator. The materials properties and boundary conditions are input into various defined region/section of the computation space. These values are used for computation within the defined boundary of the region.

The figure 3(b) is part of pre-processing that shows a typical representation of mesh generation in D.C. machine computation space generated by FEMM. This entire D.C. machine solution region defined by materials, circuit properties and boundary conditions is broken down into 4878 triangular elements and 2476 nodes before mathematical computation is carried out to obtain magnetic field distribution with the cross section of the machine.

Processing: Solving the problem by the relevant Maxwell’s equations and obtaining the field distribution in the analyzed domain of the geometry for the direct current machine at arbitrary chosen excitations and loading conditions.

Post Processing: This section of FEMM is used to view the solutions generated by FEMM solver. This is the process of calculating and presenting a D.C. machine flux pattern and deducing some results as well as parameters from the analyzed model. Figure 3(c) is the flux pattern obtained from the computations. Finite Element Method Magnetic solver that takes a set of data that describe the problem from the region and solves relevant Maxwell’s equations to obtain field values which are translated to field distribution in the analyzed domain for this machine. In order to obtain the desired result Finite Element Method Magnetic will then be run at arbitrary condition which is determine by value of the excitation.

Modelling of the DC Machine: Machine Geometrical Data:

Material Properties: The materials for this machine modeling can be user defined or can be obtained from FEMM material library.

- Yoke: 1600 Steel
- Pole Shoe: M-19 Steel
- Field Coil: Copper
- Armature Core: 1006 Steel
- Armature Winding: Copper

4.0 Results Analysis: Presented here are some of the dc Machine characteristics determine in step of post – processing when using Finite Element Method. The results are obtained using the software package FEMM version 4.0.

The magnetic flux density presented in figure 4.1 is assumed to vary sinusoidal with time around the machine cross section. Figure 4.1 shows the magnetic flux vector direction emanating from the one pole shoe section to the other forming a loop. This loop will remain fixed in the machine section while the armature core which is attached to shaft will rotate inducing voltage in the armature winding. The variation of the magnetic flux density in the air gap is shown in figure 4.2 (a) and 4.2 (b).

Magnetization curve in figure 4.3 shows a relationship between useful magnetic flux density and total ampere-turns of a typical d.c machine.

5.0 Conclusion: Computational procedure for finite element method and its application to solve magnetic field problems in d.c machine is presented. The variational formulation for Poisson’s equation, which govern the approximating function and functional minimization are presented by using first order triangular finite elements. The result obtained shows two dimensional magnetic field model of dc machine which include absolute and normal magnetic flux component in the air gap; magnetic field distribution in across section of the dc machine and its magnetization curve. The magnetization curve plotted correspond with the conventional curve obtained from typical dc machine.

Therefore FEM is an excellent tool for electromagnetic field mapping, which one could obtain electrical machine variables easily, quickly and accurately as compared to other method. Another advantage of FEM is also its ability to deal with complicated geometries such as the magnetic circuit of d.c. machines.
Fig. 3(a): Section of Computational Space

Fig. 3(b): Generated Mesh in the D.C. machine Computation Space

Fig. 3(c): Magnetic Flux Pattern by Finite Element Magnetic Method
Fig 4.1: Distribution of the magnetic flux vector in D.C. Machine Cross Section (Main pole excited only If = 0.4A p.u)

Fig 4.2(a): Air gap absolute magnetic flux density variation (Main pole excited only If = 0.4A p.u)

Fig. 4.2(b): Normal magnetic flux density component variation in the air gap (Main pole excited only If = 0.4A p.u)
Fig. 4.3: Magnetization curve of a D.C. Machine

REFERENCES