Estimation of the Unknown Parameters of the Generalized Frechet Distribution

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**Abstract:** The extreme value distribution is becoming increasingly important in engineering statistics as a suitable model to represent phenomena with usually large maximum observations. In engineering circles, this distribution is often called the Frechet model. It is one of the pioneers of extreme value statistics. The Frechet (extreme value type II) distribution is one of the probability distributions used to model extreme events. In the present study the maximum likelihood estimation of the parameters of generalized Frechet (GF) distribution are derived. Asymptotic variance covariance matrix are derived and computed numerically.

**Key words:** Fisher Information, generalized Frechet distribution, maximum likelihood estimator, sampling distribution, Pearson system.

**INTRODUCTION**

In recent years, several standard life time distributions have been generalized via exponentiation. Examples of such exponentiated distributions are the exponentiated Weibull family, the exponentiated exponential, the exponentiated Rayleigh, the exponentiated (generalized) Frechet and the exponentiated Pareto family of distributions.

Amongst the authors who have considered the exponentiated distributions are Mudholkar & Hutson. A common feature in families of exponentiated distributions is that the distribution function may be written as $F(x) = \mathcal{G}(x)^\alpha$, where $\mathcal{G}(.)$, is the distribution function of a corresponding non generalized distribution and $\alpha > 0$ denotes the generalized parameter. The generalized Frechet distribution is obtained by generalization of the Frechet distribution.

Recently, a new three parameter distribution, named as Exponentiated Frechet (EF) distribution has been introduced by Nadarajah & Kotz as a generalization of the standard Frechet distribution. There are over fifty applications ranging from accelerated life testing through to earthquakes, floods, horseracing, rain fall, queues in supermarkets, sea currents, wind speeds and track race records, see Kotz and Nadarajah.

The distribution function is

$$F(x, \alpha, \lambda, \sigma) = 1 - \left[1 - \exp\left(-\left(\frac{\sigma}{x}\right)^\lambda\right)\right]^\alpha, \quad x > 0$$

where $\alpha > 0$, $\lambda > 0$ are the shape parameters and $\sigma > 0$ is the scale parameter.

The probability density function is

$$f(x, \alpha, \lambda, \sigma) = \alpha \lambda \sigma^\lambda x^{-(\lambda+1)} \exp\left(-\left(\frac{\sigma}{x}\right)^\lambda\right) x^{-(\lambda+1)} \exp\left(-\left(\frac{\sigma}{x}\right)^\lambda\right), \quad x > 0$$

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Figure. (1) shows the probability density functions of the generalized Frechet distribution for shape parameter $\alpha = 0.25, 0.5, 1, 2$ and $2.5$; $\sigma = 1, \lambda = 1$

![Image of probability density functions of the generalized Frechet distribution for different values of shape parameter.](image)

Fig. 1: The density functions of the generalized frechet distribution for different values of shape parameter.

Its survival function is

$$S(x; \alpha, \lambda, \sigma) = \left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^4\right\}\right]^\alpha$$

The hazard rate function is an important quantity characterizing the life phenomena of the distribution. Its hazard function is

$$h(x; \alpha, \lambda, \sigma) = \frac{\alpha \lambda \sigma^4 x^{-(\alpha+1)} \exp\left\{-\left(\frac{\sigma}{x}\right)^4\right\}}{\left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^4\right\}\right]^\alpha}$$

Figure (2) illustrates some of the possible shapes of the hazard functions for selected values of shape parameters $\alpha = 0.25, 0.5, 1, 2$ and $2.5$; $\sigma = 1, \lambda = 1$

The main aim of this paper is to consider the analysis of the unknown parameters for the generalized Frechet distribution. We obtain the maximum likelihood estimators (MLE) of unknown parameters of the GF distribution. It is observed that the MLE for the unknown parameters can not be obtained in explicit from. Finally, we obtain the Fisher information matrix.

The article is arranged as follows. In section 2, the parameter estimation method is presented. In section 3, the sampling distribution for the unknown parameters is obtained by using Monte Carlo simulation. Section 4, contains the asymptotic variance covariance matrix.

2. Maximum Likelihood Estimators: In this section, the maximum likelihood estimators of GF $(\alpha, \lambda, \sigma)$ are considered in case of $\lambda$ known.

If $x_1, x_2,...,x_n$ is a random sample from the Exponentiated Frechet distribution GF $(\alpha, \lambda, \sigma)$, then the likelihood function corresponding to this sample is given by
Fig. 2: The hazard rate functions of the generalized frechet distribution for selected values of shape parameter.

\[
\prod_{i=1}^{n} f(x_i, \alpha, \lambda, \sigma) = \prod_{i=1}^{n} \left[ \alpha \lambda \sigma^\lambda \left( 1 - \exp \left\{ -\left( \frac{\sigma}{x_i} \right)^\lambda \right\} \right)^{-1} x_i^{-(\lambda+1)} \exp \left\{ -\left( \frac{\sigma}{x_i} \right)^\lambda \right\} \right] \tag{5}
\]

The log likelihood function is

\[
\ln L = n \ln (\alpha \lambda) + n \lambda \ln \sigma + (\alpha - 1) \sum_{i=1}^{n} \ln \left[ 1 - \exp \left\{ -\left( \frac{\sigma}{x_i} \right)^\lambda \right\} \right] - \left[ (1 + \lambda) \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \left( \frac{\sigma}{x_i} \right)^\lambda \right] \tag{6}
\]

When the parameter \( \lambda \) is known, we differentiate the log likelihood function with respect to \( \alpha \) and \( \sigma \) respectively to obtain the likelihood equations

Therefore

\[
\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln \left[ 1 - \exp \left\{ -\left( \frac{\sigma}{x_i} \right)^\lambda \right\} \right] = 0 \tag{7}
\]

And
Then, the maximum likelihood estimator of \( \alpha \) as a function of \( \sigma \) is obtained from (7) as

\[
\hat{\alpha}(\sigma) = \frac{-n}{\sum_{i=1}^{n} \ln \left[ 1 - \exp \left\{ -\frac{\sigma}{x_i} \right\} \right]}
\]

Substitute from (9) in (8) and solve for \( \sigma \), we obtain

\[
\frac{n}{\sigma} + (\hat{\alpha}(\sigma) - 1) \sum_{i=1}^{n} \exp \left\{ -\frac{\hat{\sigma}}{x_i} \right\} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{x_i} = 0
\]

We apply iterative procedure to find the solution of (10), once we obtain \( \hat{\sigma} \). The maximum likelihood estimators of \( \alpha \) can be obtained from (9).

3. Sampling Distributions of the Maximum Likelihood Estimators: In this section, the sampling distributions of the maximum likelihood estimators for the generalized Frechet distribution are derived using Pearson system. The Pearson system embeds seven basic types of distribution together in a single parametric framework. The selection approach is based on computing a certain quantity, \( K \), which is a function of the first four central moments, that is:

\[
K = \frac{\beta_1 \cdot (\beta_2 + 3)^2}{4 \cdot (4\beta_2 - 3) \cdot (2\beta_2 - 3\beta_1 - 6)}
\]

Where \( \beta_1 \) and \( \beta_2 \) the measures of skewness and kurtosis respectively. This value of \( K \) differs among the types of Person curves.

If \( K < 0 \), we fit type I of Persons curve, while type II is fitted if \( K = 0 \) and \( \beta_2 > 3 \). If \( K = \infty \), we obtain type III. We get type IV if \( 0 < K < 1 \). When \( K=1 \), type V is obtained. If \( K > 1 \), get type VI. Finally, if \( K = 0 \) and \( \beta_2 < 3 \) type VII is obtained.

The implementation of the Pearson system approach could be summarized in the following steps:

Step (1): A random sample \( x_1, x_2, \ldots, x_n \) from the Exponentiated Frechet was generated. Firstly a random sample \( U_{(1)}, U_{(2)}, \ldots, U_{(n)} \) of order statistics from a uniform (0,1) distribution was generated, then the \( i \)th order statistics from the EF(\( \alpha, l, \sigma \)) with \( \alpha = 1, \lambda = 1 \), and \( s = 1 \) will be obtained as follows

\[
X_{(i)} = \left[ \frac{\sigma}{\ln \left( 1 - (1 - U_{(i)})^{\frac{1}{\alpha}} \right)^{\frac{1}{l}}} \right], \quad i = 1, 2, \ldots, n
\]

Step (2): This random sample was used to estimate the unknown parameters by method of maximum likelihood mentioned in section 2.
Step (3): The mean, standard deviation, skewness, kurtosis and Pearson’s coefficient are calculated for each estimators of the unknown parameters and sample size.

As a result of computer simulation; three Pearson distributions were fitted to the maximum likelihood parameters estimators which are Pearson type I, IV and VI distributions.

Pearson type I distribution has the probability density function:

\[ P(x) = k (x - a_1)^{m_1} (a_2 - x)^{m_2} \times, \quad a_1 < x < a_2 \]

Where \( a_1 \) and \( a_2 \) are the roots of the equation \( c_n + c_1 x + c_2 x^2 = 0 \) with \( m_1 = \frac{a + a_1}{c_2 (a_2 - a_1)} \) and \( m_2 = \frac{a + a_2}{c_2 (a_2 - a_1)} \).

While Pearson type IV has the density function:

\[ P(x) = k \left[ c_0 + c_2 (x + c_1)^2 \right]^{-1} \exp \left( -\frac{a - c_1}{\sqrt{c_2 c_0}} \cdot \tan^{-1} \left( \frac{x + c_1}{\sqrt{c_2 c_0}} \right) \right), -\infty < x < \infty \]

The last one, the density function of Pearson type VI is given by:

\[ P(x) = k (x - a_1)^{m_1} (x - a_2)^{m_2}, x > a_2 \]

The sampling distributions of the maximum likelihood estimators with different sample sizes are listed in Table (1).

4. Asymptotic Variance Covariance Matrix: Fisher Information matrix \( I(\alpha, \sigma) \) can be written as follows:

\[
I(\alpha, \sigma) = -\frac{1}{n} \begin{vmatrix}
E \left( \frac{\partial^2 L}{\partial \alpha^2} \right) & E \left( \frac{\partial^2 L}{\partial \alpha \cdot \partial \sigma} \right) \\
E \left( \frac{\partial^2 L}{\partial \sigma \cdot \partial \alpha} \right) & E \left( \frac{\partial^2 L}{\partial \sigma^2} \right)
\end{vmatrix}
\]

(11)

The elements of the Fisher Information matrix can be written as

\[
E \left( \frac{\partial^2 \ln L}{\partial \alpha^2} \right) = -\frac{n}{\alpha^2}
\]

(12)
<table>
<thead>
<tr>
<th>Size</th>
<th>n</th>
<th>σ</th>
<th>μ</th>
<th>Var.</th>
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<th>skewness</th>
<th>kurtosis</th>
<th>Person’s coefficient</th>
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<td>0.502</td>
<td>3.333</td>
<td>-0.58</td>
<td>I</td>
<td></td>
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</table>

\[
E \left( \frac{\partial^2 \ln L}{ \partial \alpha \partial \sigma} \right) = E \left( \frac{n}{\sigma^2} \sum_{i=1}^{n} \frac{1}{x_i} \exp \left\{ - \frac{\sigma}{x_i} \right\} \right)
\]

And

\[
E \left( \frac{\partial^2 \ln L}{ \partial \sigma^2} \right) = E \left( \frac{-n}{\sigma^2} - (\alpha - 1) \sum_{i=1}^{n} \frac{1}{x_i^2} \exp \left\{ - \frac{\sigma}{x_i} \right\} \right)
\]
to compute (13) and (14) we obtain explicit expression of expectation of the forms.

\[
E \left( \frac{1}{x_i} \cdot \exp \left( - \frac{\sigma}{x_i} \right) \right) \quad \text{and} \quad E \left( \frac{1}{x_i} \cdot \exp \left( - \frac{\sigma}{x_i} \right) \right)^2
\]

for \( i=1,2, \ldots, n \)

Since the density of the \( \hat{p} \) order statistic from random sample of size \( n \) following the EF\((\alpha,1,\sigma)\) distribution

\[
f_{x_i}(x) = \frac{n!}{(i-1)!(n-i)!} \alpha \sigma x_i^{-2} \left[ \exp \left( \frac{\sigma}{x_i} \right) \right]^{i-1} \left[ 1 - \exp \left( \frac{\sigma}{x_i} \right) \right]^{n-i-1}
\]

Then

\[
E \left( \frac{1}{x_i} \cdot \exp \left( - \frac{\sigma}{x_i} \right) \right) = \int_0^\infty \frac{1}{x_i} \cdot \exp \left( - \frac{\sigma}{x_i} \right) f_{x_i}(x) \, dx
\]

\[
= \frac{n! \alpha \sigma}{(i-1)!(n-i)!} \int_0^\infty x_i^{-3} \left( \exp \left( \frac{\sigma}{x_i} \right) \right)^{i-2} \left[ 1 - \exp \left( \frac{\sigma}{x_i} \right) \right]^{-(n-i+1)} \, dx
\]

\[
x_i = \int_0^{\infty} x_i^{-3} \left( \exp \left( \frac{\sigma}{x_i} \right) \right)^{i-2} \left[ 1 - \exp \left( \frac{\sigma}{x_i} \right) \right]^{-(n-i+1)} \, dx
\]

If put \( y = \exp \left( - \frac{\sigma}{x_i} \right) \)
\[ x = \sigma \cdot \left( \log \left( \frac{1}{y} \right) \right)^{-1}, \quad dx = \sigma \cdot \left( \log \left( \frac{1}{y} \right) \right)^{-2} \cdot y^{-1} \cdot dy; \quad x = 0 \iff y = 0 \text{ and } x = \infty \iff y = 1 \]

\[ I = \int_0^{1} \left( \sigma \left( \log \left( \frac{1}{y} \right) \right) \right)^{-3} \cdot y^2 \cdot \left[ 1 - y \right]^{\alpha(n-i+1)-2} \cdot \left[ 1 - \left( 1 - y \right) \alpha \right]^{i-1} \cdot \left( \log \left( \frac{1}{y} \right) \right)^{-2} \cdot y^{-1} \cdot dy \]

\[ = \frac{1}{\sigma^2} \int_0^{1} \left( \log \left( \frac{1}{y} \right) \right) \cdot y \cdot \left[ 1 - y \right]^{\alpha(n-i+1)-2} \cdot \left[ 1 - \left( 1 - y \right) \alpha \right]^{i-1} \cdot dy \]

\[ = \frac{1}{\sigma^2} \sum_{k=0}^{i-1} \binom{i-1}{k} \cdot (-1)^{(k+1)} \cdot \int_0^{1} \left( \log \left( \frac{1}{y} \right) \right) \cdot y \cdot \left[ 1 - y \right]^{\alpha(n-i+1)-2} \cdot \left[ 1 - \left( 1 - y \right) \alpha \right]^{(i-k)-1} \cdot dy \]

\[ = \frac{1}{\sigma^2} \sum_{k=0}^{i-1} \binom{i-1}{k} \cdot (-1)^{(k+1)} \cdot \frac{1}{\sigma^2} \sum_{j=0}^{\alpha(n-k)-2} \left( \log \left( \frac{1}{y} \right) \right) \cdot y^{(\alpha(n-k)-(j+1))} \cdot \phi \]

Thus we can use the integral

\[ \frac{1}{\sigma^2} \sum_{k=0}^{i-1} \binom{i-1}{k} \cdot \frac{1}{\sigma^2} \sum_{j=0}^{\alpha(n-k)-2} \left( \log \left( \frac{1}{y} \right) \right) \cdot y^{(\alpha(n-k)-(j+1))} \cdot \phi \]

When \( \mu = 2, \nu = \alpha(n-k)-j \) and \( x = y \) Thus

\[ \frac{1}{\sigma^2} \sum_{k=0}^{i-1} \binom{i-1}{k} \cdot \frac{1}{\sigma^2} \sum_{j=0}^{\alpha(n-k)-2} \left( \log \left( \frac{1}{y} \right) \right) \cdot y^{(\alpha(n-k)-(j+1))} \cdot \phi = \frac{1}{(\alpha(n-k)-j)^2} \cdot \Gamma(2) = \frac{1}{(\alpha(n-k)-j)^2} \]

Therefore
Similarly to compute

\[
E \left( \frac{1}{x_i} \right)^2 \cdot \exp \left\{ - \frac{\sigma}{x_i} \right\} \right)^2 \left( 1 - \exp \left\{ - \frac{\sigma}{x_i} \right\} \right) = \frac{\alpha^2 \sigma^2}{(n-i)! (n-i)!} \cdot \left( \frac{1}{x_i} \right)^4 \cdot \left( 1 - \exp \left\{ - \frac{\sigma}{x_i} \right\} \right)^2 \int_{x_i} \left( \frac{1}{x} \right) dx
\]

In the same way we use \( y = \exp \left\{ - \frac{\sigma}{x} \right\} \), the above equation reduce to

\[
= \frac{n! \sigma^2 \alpha}{(n-i)! (n-i)!} \cdot \sum_{K=0}^{i-1} \frac{(-1)^{i-K}}{i-K} \cdot \left( \frac{\alpha(n-k)-3}{n-k} \right)^{i-K} \cdot \left( \frac{\alpha(n-k)}{n-k} \right)^{i-K} \cdot \left( \frac{\alpha(n-k) - (j+1)}{n-k} \right) \] 

Then by using (15) when \( \mu = 3 \), \( v = \alpha(n-k) - (j+1) \) and \( x = y \).
Now, we compute the elements of Fisher information matrix for the generalized Frechet distribution numerically. For different sample size $n = 10, 15, 20, 25, 30, 35, 40, 45, \text{ and } 50$. Compute $I(\lambda, \sigma)$ which defined in (11).

We compute

$$\frac{1}{n} \mathbb{E} \left( \frac{\partial^2}{\partial \alpha^2} \right) \quad \frac{1}{n} \mathbb{E} \left( \frac{\partial^2}{\partial \alpha \cdot \partial \sigma} \right) \quad \frac{1}{n} \mathbb{E} \left( \frac{\partial^2}{\partial \sigma^2} \right)$$

which corresponding to MLE, of the parameters $(\alpha, \sigma)$ of the generalized Frechet distribution in each sample size. The results are listed at Table (2).

**Table 2**: Elements of the approximate Fisher information matrices of \(\text{EF}(\alpha, \lambda, \sigma)\)

<table>
<thead>
<tr>
<th>Sample Size $n$</th>
<th>MSEs of Parameter</th>
<th>Var((\alpha),(\alpha))</th>
<th>Cov((\alpha),(\sigma))</th>
<th>Var((\sigma),(\sigma))</th>
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<td></td>
<td>(\hat{\sigma})</td>
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<td></td>
<td>(\hat{\sigma})</td>
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<td>(\hat{\sigma})</td>
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<tr>
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REFERENCES