On Finite Mixture of two-Component Exponentiated Gamma Distribution

A.I. Shawky and R.A. Bakoban

1King Abdulaziz University, Faculty of Sciences, Department of Statistics, P.O. Box 80203, Jeddah 21589, Saudi Arabia.
2King Abdulaziz University, Faculty of Sciences, Branch Girls, Department of mathematics, P.O. Box 4269, Jeddah 21491, Saudi Arabia.

Abstract: This article is considered with the problem of estimating the parameters, reliability and failure rate functions of the finite mixture of two components from exponentiated gamma distributions. The maximum likelihood and Bayes methods of estimation are used. An approximation form due to Lindley (1980) is used for obtaining the Bayes estimates under the squared error loss and LINEX loss functions. Comparisons are made between these estimators and the maximum likelihood ones using Monte Carlo simulation study.

Key words: Finite mixture; Statistical properties; Maximum likelihood; Bayesian analysis; Squared error loss function; LINEX loss function; Lindley's approximation; Exponentiated gamma distribution.

INTRODUCTION

One of the important families of distributions in lifetime tests is the exponentiated gamma (EG) distribution with probability density function (p.d.f.)

$$f(x; \theta) = \theta x e^{-\theta x} \{1-e^{-\theta x (x+1)}\}^{\theta-1},$$

$$x > 0, \theta > 0,$$  \hspace{1cm}  (1.1)

and the cumulative distribution function (c.d.f.) is given by

$$F(x; \theta) = \left[1-e^{-\theta x (x+1)}\right]^{\theta},$$

$$x > 0, \theta > 0,$$  \hspace{1cm}  (1.2)

see Shawky and Bakoban [30-34].

When the shape parameter $\theta = 1$ in both (1.1) and (1.2) give the p.d.f. and c.d.f. of gamma distribution with shape parameter $\alpha = 2$ and scale parameter $\beta = 1$, i.e., $G(2,1)$.

Mixtures of life distributions occur when two different causes of failure are present, each with the same parametric form of life distributions. In recent years, the finite mixtures of life distributions have proved to be of considerable interest both in terms of their methodological development and practical applications [22,19,23,21,7]. On characterization of mixtures were studied by, Nassar and Mahmoud [21], Nassar [24], Gharib [11] and Ismail and El Khodary [14]. Many authors interested in inference on mixtures of exponential distributions among them Rider [30], Everitt and Hand [9], Al-Hussaini [5], Bartoszewicz [4] and Jaheen [15]. Radhakrishna et al. [27] derived moments and maximum likelihood estimators of the parameters of two component mixture generalized gamma distribution. Also, Ahmad et al. [1] derive approximate Bayes estimation for mixtures of two Weibull distributions under type II censoring. On finite mixture of two component Gompertz distribution considered by Al-Hussaini et al. [5]. Several papers discussed normal mixtures, for example, Hosmer and Holgersson and Jomer [12]. Moreover, John [6] briefly outlines the use of the methods of moments and maximum likelihood in estimating the parameters of two component gamma mixtures. Further, a mixture of two gamma distributions applied to rheumatoid arthritis was discussed by Masuyama [28]. Also, Gharib [10] obtained two characterizations of a gamma mixture distribution. Further, a mixture of two inverse Weibull distribution was studied by Sultan et al. [30]. Also, Elsherpien [6] estimated the parameters of mixed generalized exponentionaly distributions.

A random variable $T$ is said to follow a finite mixture distribution with $k$ components, if the p.d.f. of $T$ can be written in the form Titterington et al. [30]

$$f(t) = \sum_{j=1}^{k} p_j f_j(t),$$

where, for $j = 1, \ldots, k, f_j(t)$ the $j^{th}$ p.d.f.
component and the mixing proportions, \( p_j \), satisfy the conditions \( 0 < p_j < 1 \) and \( \sum_{j=1}^{k} p_j = 1 \). The corresponding c.d.f. is given by

\[
F(t) = \sum_{j=1}^{k} p_j F_j(t),
\]

where \( F_j(t) \) is the \( j^{th} \) c.d.f. component. The reliability function (RF) of the mixture is given by

\[
R(t) = \sum_{j=1}^{k} p_j R_j(t),
\]

where \( R_j(t) \) is the \( j^{th} \) reliability component.

The hazard function (HF) of the mixture is given by

\[
H(t) = \frac{f(t)}{R(t)},
\]

where \( f(t) \) and \( R(t) \) are defined in (1.3) and (1.5), respectively, i.e.

\[
H(t) = \frac{\sum_{j=1}^{k} p_j f_j(t)}{\sum_{j=1}^{k} p_j R_j(t)}.
\]

A mixture is identifiable if there exist a one-to-one correspondence between the mixing distribution and a resulting mixture. That is, there is a unique characterization of the mixture.

A class \( D \) of a mixture is said to be identifiable if and only if, for all \( f(t) \in D \) the equality of the two representations\(^{(39)}\)

\[
\sum_{i=1}^{\ell} p_i f_i(t \mid \theta_i) = \sum_{j=1}^{\ell'} p'_j f_j(t \mid \theta'_j),
\]

implies that \( C = C' \) and for all \( i \) there exists some \( j \) such that \( p_i = p'_j \) and \( \theta_i = \theta'_j \). Teicher\(^{(38)}\) has studied the class of gamma mixtures and proved its identifiable. We now show the identifiability of a mixture of \( k \) EG components in the following theorem.

**Theorem 1:** A finite mixture of \( k \) exponentiated gamma components is identifiable.

**Proof:** Teicher\(^{(37)}\) showed that a finite mixture of exponential components is identifiable. If \( Y \sim \text{Exp}(\theta) \) and \( e^{-\theta} = 1 - e^{-\theta}(T + 1) \), follows that \( T \sim \text{EG}(\theta) \) The transformation is one-to-one and onto, so a finite mixture of \( \text{EG}(\theta_j), j=1, 2, ..., k \) components is identifiable. It follows that

\[
\sum_{j=1}^{k} p_i [1 - e^{-\theta_j}(t + 1)]^{\theta_j'} = \sum_{j=1}^{k} p'_i [1 - e^{-\theta_j}(t + 1)]^{\theta_j'},
\]

implies that \( k = k' \) and for all \( i \), there exists some \( j \) such that \( p_i = p'_j \) and \( \theta_i = \theta'_j \). Therefore, a finite mixture of \( k \) \( \text{EG}(\theta_j), j=1, 2, ..., k \) components is identifiable.

**Mixture of \( k \) EG Components:** By indexing the p.d.f. (1.1) and c.d.f. (1.2), \( j=1, 2, ..., k \) then substituting in (1.3) and (1.4), the p.d.f. and c.d.f. of a finite mixture of \( k \) \( \text{EG}(\theta_j), j=1, 2, ..., k \) components are given, respectively, by

\[
f(t) = \sum_{j=1}^{k} p_j \theta_j t e^{-\theta_j[t + 1]}[1 - e^{-\theta_j(t + 1)}]^{\theta_j - 1},
\]

\( t > 0, \theta_j > 0 \),

(1.7)

and

\[
F(t) = \sum_{j=1}^{k} p_j [1 - e^{-\theta_j(t + 1)}]^{\theta_j},
\]

\( t > 0, \theta_j > 0 \),

(1.8)

where, for \( j=1, 2, ..., k \), \( 0 < p_j < 1 \) and
By observing that \( R(t) = 1 - F(t) \) and \( \sum_{j=1}^{k} p_j = 1 \), he RF of a mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \) components can obtained from (1.5) and (1.8) as

\[
R(t) = \sum_{j=1}^{k} p_j \{1 - [1 - e^{-\theta_j (t+1)}]\}^\theta_j, \quad t > 0, \theta_j > 0. \tag{1.9}
\]

Dividing (1.7) by (1.9), we obtain the HF of a mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \) components as

\[
H(t) = \frac{\sum_{j=1}^{k} p_j \theta_j t e^{-\theta_j [1 - e^{-\theta_j (t+1)}]} \theta_j^{-1}}{\sum_{j=1}^{k} p_j \{1 - [1 - e^{-\theta_j (t+1)}]\}^\theta_j}. \tag{1.10}
\]

If \( k=2 \), the p.d.f., c.d.f., RF and HF of a mixture of \( EG(\theta_j), j = 1, 2 \) components are then given, respectively, by

\[
f(t) = p \theta_1 t e^{-t} [1 - e^{-t (t+1)}] \theta_1^{-1} + (1 - p) \theta_2 t e^{-t} [1 - e^{-t (t+1)}] \theta_2^{-1}, \tag{1.11}
\]

\[
F(t) = p [1 - e^{-t (t+1)}] \theta_1 + (1 - p) [1 - e^{-t (t+1)}] \theta_2, \tag{1.12}
\]

\[
R(t) = p \{1 - [1 - e^{-t (t+1)}]\} \theta_1 + (1 - p) \{1 - [1 - e^{-t (t+1)}]\} \theta_2, \tag{1.13}
\]

\[
H(t) = \frac{f(t)}{R(t)}. \tag{1.14}
\]

where \( f(t) \) and \( R(t) \) are given, respectively, by (1.11) and (1.13).

This article is considered with finite mixture of exponentiated gamma distributions. Statistical properties for finite mixture of \( k \) exponentiated gamma components are derived in section 2. Maximum likelihood estimators of the two shape parameters, reliability and failure rate functions of a mixture of two exponentiated gamma distributions are derived from complete and type II censored samples in section 3. In section 4, Bayes estimators of the two shape parameters, reliability and failure rate functions of a mixture of two exponentiated gamma distributions are derived under the squared error loss and LINEX loss functions. Also, Monte Carlo simulation study are made in section 5. Finally, concluding remarks about comparisons between these estimators and the maximum likelihood ones are considered in section 6.

2. Statistical Properties:
2.1 Moments and Some Measures: The \( p^{th} \) moment about the origin, \( \mu_p = E(T^p) \) of a mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \) components with p.d.f. (1.7) in the non-closed form is
\[ \mu_r = \sum_{j=1}^{k} p_j \theta_j j! \int_0^\infty \theta^{-r+1} \left( 1 - e^{-t} - t e^{-t} \right) \theta^{-1} dt, \quad r = 0, 1, 2, \ldots \] (2.1)

that is, for positive real value of \( \theta_j \), \( \mu_r \) takes the closed form

\[ \mu_r = \sum_{j=1}^{k} \sum_{\delta=0}^{\delta_j} \sum_{\kappa=0}^{\kappa_j} p_j (-1)^\delta \binom{\theta-1}{\delta_j} \binom{j_1}{\delta_1} \binom{j_2}{\kappa_1} \frac{\Gamma(r + k_1 + 2)}{(j_1 + 1)^{r + \kappa + 2}}, \]

(2.2)

where

\[ A_r(\theta_j) = \sum_{\delta=0}^{\delta_j} \sum_{\kappa=0}^{\kappa_j} (-1)^\delta \binom{\theta-1}{\delta_j} \binom{j_1}{\delta_1} \binom{j_2}{\kappa_1} \frac{\Gamma(r + k_1 + 2)}{(j_1 + 1)^{r + \kappa + 2}}, \] (2.3)

\[
\binom{v}{j} = \binom{v}{j} / j!, \quad j > 0, \quad v \text{ is a real number} \\
\binom{v}{0} = 1, \quad j = 0,
\]

where

\[ (v)_{(j)} = v (v - 1) \ldots (v - j + 1), \quad j \geq 1 \quad \text{and} \quad (v)_{(0)} = 1. \]

When \( k = 2 \), the \( r^{th} \) moment about the origin, \( \mu_r = \mathbb{E}(T^r) \) of a mixture of two \( EG(\theta_j) \), \( j = 1, 2, \ldots \), components with p.d.f. (1.11) is given by

\[ \mu_r = p \theta_1 [\Gamma(r + 2) + A_r(\theta_1)] + (1 - p) \theta_2 [\Gamma(r + 2) + A_r(\theta_2)]. \]

The above closed form of \( \mu_r \) allows us to derive the following forms of statistical measures for the mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \) components:

1- Coefficient of variation:

\[ CV = \frac{\sigma}{\mu} = \sqrt{\frac{\sum_{j=1}^{k} p_j \theta_j (6 + A_2) - \left( \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right)^2}{\sum_{j=1}^{k} p_j \theta_j (2 + A_1)}}. \] (2.5)

2- Skewness:
\[ v_3 = \frac{\sum_{j=1}^{k} p_j \theta_j (24 + A_3) - 3 \left( \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right) \left( \sum_{j=1}^{k} p_j \theta_j (6 + A_2) \right) + 2 \left( \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right)^3}{\left[ \sum_{j=1}^{k} p_j \theta_j (6 + A_2) - \left( \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right)^2 \right]^{3/2}}. \] (2.6)

3- Kurtosis:

\[ v_4 = \frac{\sum_{j=1}^{k} p_j \theta_j (120 + A_4) - 4 \left( \sum_{j=1}^{k} p_j \theta_j (24 + A_3) \right) \left[ \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right] + W}{\left[ \sum_{j=1}^{k} p_j \theta_j (6 + A_2) - \left( \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right)^2 \right]^{2}}, \] (2.7)

where

\[ W = \left\{ \sum_{j=1}^{k} p_j \theta_j (6 + A_2) \right\} \left\{ \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right\}^2 - 3 \left\{ \sum_{j=1}^{k} p_j \theta_j (2 + A_1) \right\}^4, \]

where abbreviation

\[ A_r \text{ is for } A_r \left( \theta_j \right), \quad r = 0, 1, 2, \ldots \]

The result developed above can be derived from the moment generating function (m.g.f.),

\[ M_X(t) = E \left( e^{tX} \right) \text{ for the mixture of } E \left( \theta_j, \ j=1, 2, \ldots, k \right), \text{ components as follows.} \]

\[ M_X(t) = \sum_{j=1}^{k} p_j \theta_j \int_0^\infty x e^{-(1-r)x} \left( 1 - e^{-x} - xe^{-x} \right)^{\theta_j - 1} dx, \quad t > 0 \]

\[ = \sum_{j=1}^{k} \sum_{a=0}^{\theta_j-1} \sum_{b=0}^{a} p_j \theta_j \left( -1 \right)^a \binom{\theta_j - 1}{a} \binom{a}{b} \frac{\Gamma(k_1 + 2)}{(1 - t + j_1)^{k_1 + 2}}. \] (2.8)

When \( p_1 = 1, 2, 8 \) gives the m.g.f. of \( E \) distribution with p.d.f. given in (1.1).

The mean deviation, \( M.D. = E \left( |X - \mu| \right) \) of the mixture of \( E \left( \theta_j, \ j=1, 2, \ldots, k \right), \text{ components,} \)

random variable \( X \) with p.d.f. (1.7) in the non-closed form is

\[ M.D. = 2 \int_0^\infty F(x) dx, \] (2.9)

where

\[ \mu = \mu_1 \text{ is defined in (2.2) with } r = 1, \text{ and } F(x) \text{ is the c.d.f. (1.8).} \]

that is, for positive real value of \( \theta_j, M.D. \text{ takes the closed form} \]
\[ M. D. = 2 \mu + 2 \sum_{j=1}^{k} \sum_{k_1=0}^{\infty} \sum_{k_1=0}^{\infty} p_j (-1)^{\delta} \left( \frac{\theta_j}{j_1} \right) \left( \frac{\theta_j}{k_1} \right) e^{-\lambda_{\mu} k_1} \frac{\Gamma(k_1 + 1)}{j_1^{k_1 + 1}} \frac{(j_1 \mu)^l}{l!} \] (2.10)

**Median and Mode:** The median of the mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \), components cannot be found in explicit form.

We derive the median \( m \) as the numerical solution of the following equation:

\[ \sum_{j=1}^{k} \sum_{k_1=0}^{\infty} \sum_{k_1=0}^{\infty} (-1)^{\delta} p_j \left( \frac{\theta_j}{j_1} \right) \left( \frac{\theta_j}{k_1} \right) e^{-\lambda_{\mu} k_1} m_{k_1} = 0.5 \] (2.11)

where \( \theta_j \) is a real number.

Next, to find the mode for the mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \), components, we differentiate \( f(t) \) with respect to \( t \), so (1.7) gives

\[ f''(t) = \frac{k}{t} \sum_{j=1}^{k} p_j \left[ \left( \frac{1}{t} - 1 \right) t e^{-t} \left( 1 - e^{-t} - t e^{-t} \right) \right] f_j(t). \] (2.12)

Then by equating (2.12) with zero, we get mode(s). We observe that the mixture of \( EG(\theta_j), j = 1, 2, \ldots, k \), components, may be unimodal (see Fig. 1) or bimodal (see Fig. 2) with mode(s) can be found numerically by solving (2.12). Figure 1 shows some densities of \( EG(\theta_j), j = 1, 2 \), components and their unimodal mixtures.

![Fig. 1](image)

**Fig. 1:** Shapes of \( EG(\theta_j), j = 1, 2 \), components and their mixtures with \( (p \theta_1 \theta_2) \).
3. Maximum Likelihood Estimation: Suppose a Type-II censored sample $\mathbf{T} = (T_{1n}, T_{2n}, \ldots, T_{rn})$ where $T_{in}$ is the time of the $i^{th}$ component to fail. This sample of failure times are obtained and recorded from a life test of $n$ items whose life times have the mixture of $EG(\Theta_j)$, $j = 1, 2$, components with p.d.f. and c.d.f. given,

respectively, by (1.11) and (1.12). The likelihood function in this case can be written as:

$$L(\Theta | \mathbf{T}) = \frac{n!}{(n-r)!} \left[ \prod_{i=1}^{r} f(t_{in}) \right] [R(t_{rn})]^{n-r},$$

(3.1)

where $f(t)$ and $R(t)$ are given, respectively, by (1.11) and (1.13).

Figure 2 shows some densities of $EG(\Theta_j)$, $j = 1, 2$, components and their bimodal mixtures.

The natural logarithm of the likelihood function (3.1) is given by

$$l = \ln L(\Theta | \mathbf{T}) = \ln \left( \frac{n!}{(n-r)!} \right) + \sum_{i=1}^{r} \ln f(t_{in}) + (n-r) \ln R(t_{rn}).$$

(3.2)

Assuming that the parameters, $\Theta_1$ and $\Theta_2$, are unknown, the likelihood equations are given, for $j = 1, 2$, by

$$l_j = \frac{\partial l}{\partial \Theta_j} = \sum_{i=1}^{r} \left[ \frac{1}{f(t_{in})} \frac{\partial f(t_{in})}{\partial \Theta_j} \right] + \frac{n-r}{R(t_{rn})} \frac{\partial R(t_{rn})}{\partial \Theta_j} = 0.$$  

(3.3)

From (1.11) and (1.13), respectively, we have

$$\frac{\partial f(t_{i:n})}{\partial \Theta_j} = p_j f_j(t_{i:n}, \Theta_j) \kappa_j(t_{i:n})$$

(3.4)
and
\[
\frac{\partial R(t_r:n)}{\partial \theta_j} = -p_j F_j(t_r:n) \alpha(t_r:n),
\]
where \( p_1 = p, p_2 = 1 - p \),
\[
\kappa_j(t_i:n) = \theta_j^{-1} + \alpha(t_i:n),
\]
and
\[
\alpha(t_i:n) = \ln[1 - e^{-t_i:n}(t_i:n+1)].
\]
Substituting (3.4) and (3.5) in (3.3), we obtain
\[
l_j = p_j \left\{ \sum_{i=1}^{n} \xi_j(t_i:n) - (n-j)\xi_j(t_r:n)\alpha(t_r:n) \right\} = 0,
\]
where, for \( j = 1, 2 \), and \( i = 1, \ldots, r \),
\[
\xi_j(t_i:n) = \frac{f_j(t_i:n)}{f(t_i:n)} , \quad \xi_j^*(t_r:n) = \frac{F_j(t_r:n)}{R(t_r:n)},
\]
and \( \kappa_j(t_i:n), \alpha(t_i:n) \) are given by (3.6) and (3.7), respectively.

The solution of the two nonlinear likelihood equations (3.8) yields the maximum likelihood estimate
(\text{MLE}) \( \hat{\theta} = (\hat{\theta}_{1,M}, \hat{\theta}_{2,M}) \) if \( \theta = (\theta_1, \theta_2) \).

The MLE's of \( R(t) \) and \( H(t) \) are given, respectively, by (1.13) and (1.14) after replacing \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) by their corresponding MLE's, \( \hat{\theta}_{1,M} \) and \( \hat{\theta}_{2,M} \).

4. Bayes Estimation: Let \( \Theta_1 \) and \( \Theta_2 \) be independent random variables. The joint prior density of the random vector \( \Theta = (\Theta_1, \Theta_2) \) is thus given by
\[
g(\Theta) = g(\theta_1, \theta_2) = g_1(\theta_1)g_2(\theta_2),
\]
where, for \( j = 1, 2 \), \( g_j(\theta_j) \) is a prior density function of \( \Theta_j \) and \( \theta_j > 0 \).

We choose the random variable \( \Theta_j, j = 1, 2 \), follow gamma distribution with shape parameter \( \alpha_j \), and scale parameter \( \beta = 1 \) i.e., \( G(\alpha, \beta) \). Its density function is
\[
g_j(\theta_j) = \frac{\theta_j^{\alpha_j-1}e^{-\theta_j}}{\Gamma(\alpha_j)}, \quad \theta_j > 0, \quad \alpha_j > 0.
\]

Based on the above considerations, the prior density function of \( \Theta \) is given by
\[
g(\Theta) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \sum_{-1}^{\alpha_2} e^{-\theta_j} \theta_j^{\alpha_j-1}, \quad \theta_j > 0, \quad j = 1, 2.
\]

It is well known that the posterior density function of \( \Theta \) given the observations, denoted by \( q(\Theta|\xi) \), is given by
\[
q(\Theta|\xi) = \frac{L(\Theta|\xi)g(\Theta)}{\int_{\Theta} L(\Theta|\xi)g(\Theta) d\Theta}.
\]

It then follows that the Bayes estimator \( \bar{\phi} \) of a function of the parameter \( \phi(\Theta) \) is given by the ratio
\[ \tilde{\phi} = \mathbb{E}[\phi(\theta) | \mathcal{I} = \xi] = \int_{\Omega} \phi(\theta) g(\theta | \xi) d\theta \]
\[ = \frac{\int_{\Omega} \phi(\theta) L(\theta | \xi) g(\theta) d\theta}{\int_{\Omega} L(\theta | \xi) g(\theta) d\theta}, \quad (4.3) \]
where \( L(\theta | \xi) \) is given by (3.1), \( g(\theta) \) by (4.1) and \( \Omega \) is the region in the \( \Theta_1 \Theta_2 \) plane on which the posterior density \( g(\theta | \xi) \) is positive.

The ratio of the integrals (4.3) may thus be approximated by using a form due to Lindley\(^{[18]} \) which reduces, in the case of two parameters, to the form

\[ \tilde{\phi} = \frac{\phi(\theta)}{2} + \rho_1 \phi_{12} + \rho_2 \phi_{21} + \frac{1}{2}[l^{*}_{30} \phi_{12} + l^{*}_{21} \phi_{21} + l^{*}_{12} \phi_{21} + l^{*}_{03} \phi_{21}], \quad (4.4) \]
where \( \theta = (\theta_1, \theta_2), S = \sum_{i=1}^{2} \sum_{j=1}^{2} \phi_{ij} \sigma_{ij} = (i, j)th \) element in the matrix \( \Sigma \), where \( \Sigma = -[J(\theta)]^{-1}, \quad J(\theta) = [J_{ij}], J_{ij} = \frac{\partial^2 I}{\partial \theta_i \partial \theta_j} = \ln L(\theta | \xi), L(\theta | \xi) \) given by

For \( i \neq j, \phi_{ij} = \phi_1 \sigma_{ij} + \phi_2 \sigma_{ij}, \psi_{ij} = (\phi_1 \sigma_{ij} + \phi_2 \sigma_{ij}) \sigma_{ij}, c_{ij} = 3 \phi_1 \sigma_{ij} \sigma_{ij} + \phi_2 (\sigma_{ij} \sigma_{ij} + 2 \sigma_{ij}^2), \)
where \( \phi_i = \frac{\partial \phi}{\partial \theta_i} \) and \( \rho_i = \frac{\partial \rho}{\partial \theta_i}, \rho = \ln[g(\theta)], g(\theta) \) given in (4.1). Finally,

\[ l^{*}_{30} = \frac{\partial l_{11}}{\partial \theta_1}, l^{*}_{21} = \frac{\partial l_{12}}{\partial \theta_1}, l^{*}_{12} = \frac{\partial l_{12}}{\partial \theta_2}, \quad l^{*}_{03} = \frac{\partial l_{22}}{\partial \theta_2}. \]

Now, we apply Lindley's form (4.4), we first obtain the elements \( \sigma_{ij} \), which can be obtained as

\[ \sigma_{11} = -\frac{l_{22}}{D}, \quad \sigma_{22} = -\frac{l_{11}}{D}, \quad \sigma_{12} = \sigma_{21} = \frac{l_{12}}{D}, \quad (4.5) \]
where \( D = l_{11}l_{22} - l_{12}^2, \quad (4.6) \]

\[ l_{12} = l_{21} = -p_1 p_2 \left\{ \sum_{i=1}^{r} \varphi(t_{in}) + (n-r) a^2 (t_{r,n}) v \mu(t_{r,n}) \right\}, \quad (4.7) \]
and for \( i = 1, 2, \ldots, r, \)

\[ \varphi(t_{in}) = \kappa_1 (t_{in}) \kappa_2 (t_{in}) \xi_1 (t_{in}) \xi_2 (t_{in}), \quad (4.8) \]

\[ \psi(t_{r,n}) = \xi_1^* (t_{r,n}) \xi_2^* (t_{r,n}), \quad (4.9) \]
\[ l_{j j} = -p_j \left\{ \sum_{i=1}^{r} A_j(t_{in}) + (n - r) \omega(t_{rn}) \right\} B_j(t_{rn}), \quad j = 1, 2, \quad s = 1, 2, \quad j \neq s, \] 

(4.10)

\[ A_j(t_{in}) = \xi_j(t_{in}) \theta_j^2 - p_j \xi_j^2(t_{in}) \xi_j(t_{in}) \xi_j(t_{in}), \] 

(4.11)

\[ B_j(t_{rn}) = \tau_j(t_{rn}) + p_j \xi_j^2(t_{rn}) \omega(t_{rn}). \] 

(4.12)

For \( j = 1, 2 \), the functions \( \tau_j(.) \) and \( \omega(.) \) are given by (3.6) and (3.7), \( \xi_j(.) \) and \( \xi_j^*(.) \) by (3.9) and \( \tau_j(.) \) is given by

\[ \tau_j(t_{rn}) = \frac{f_j(t_{rn})}{R(t_{rn})}. \] 

(4.13)

Furthermore,

\[ l_{30}^* = \frac{\partial l_{11}}{\partial \theta_1} = -p_1 \left\{ \sum_{i=1}^{r} \frac{\partial A_1(t_{in})}{\partial \theta_1} + (n - r) \omega(t_{rn}) \frac{\partial B_1(t_{rn})}{\partial \theta_1} \right\}, \] 

(4.14)

\[ l_{03}^* = \frac{\partial l_{22}}{\partial \theta_2} = -p_2 \left\{ \sum_{i=1}^{r} \frac{\partial A_2(t_{in})}{\partial \theta_2} + (n - r) \omega(t_{rn}) \frac{\partial B_2(t_{rn})}{\partial \theta_2} \right\}, \] 

(4.15)

\[ l_{21}^* = \frac{\partial l_{21}}{\partial \theta_1} = -p_1 p_2 \left\{ \sum_{i=1}^{r} \frac{\partial \varphi(t_{in})}{\partial \theta_1} + (n - r) \omega(t_{rn}) \frac{\partial \varphi(t_{rn})}{\partial \theta_1} \right\}, \] 

(4.16)

where, for \( j, s = 1, 2 \) and \( j \neq s \),

\[ \frac{\partial A_j(t_{rn})}{\partial \theta_j} = \theta_j^2 \xi_j(t_{in}) - 2 \theta_j^3 \xi_j(t_{in}) - p_j \xi_j^2(t_{in}) \xi_j(t_{in}) \xi_j(t_{in}), \]

\[ \frac{\partial B_j(t_{rn})}{\partial \theta_j} = \tau_j'(t_{rn}) + 2 p_j \xi_j^2(t_{rn}) \omega(t_{rn}) \xi_j(t_{rn}), \]

\[ \frac{\partial \varphi(t_{rn})}{\partial \theta_j} = \kappa_j(t_{in}) \xi_j(t_{in}) \left\{ \kappa_j(t_{in}) \xi_j(t_{in}) + \xi_j(t_{in}) \kappa_j(t_{in}) \right\}, \]
In Bayesian estimation, we consider two types of loss functions. The first is the squared error loss function (quadratic loss) which is classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the LINEX (linear-exponential) loss function which is asymmetric, was introduced by Varian. These loss functions were widely used by several authors; among of them Rojo, Basu and Ebrahimi, Pandey, and Soliman.

The quadratic loss for Bayes estimate of a parameter \( \beta \), say, is the posterior mean assuming that exists, denoted by \( \hat{\beta}_s \). The LINEX loss function may be expressed as
\[
L(\Delta) \propto e^{c\Delta} - c\Delta - 1, \quad c \neq 0,
\]
where \( \Delta = \hat{\beta} - \beta \). The sign and magnitude of the shape parameter \( c \) reflects the direction and degree of asymmetry respectively. (If \( c > 0 \), the overestimation is more serious than underestimation, and vice-versa.) For \( c \) closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function Equation (4.22) is
\[
E_{\beta} [L(\hat{\beta} - \beta)] \propto \exp(c\hat{\beta}) [\exp(-c\beta)] - c(\hat{\beta} - E_{\beta}(\beta)) - 1,
\]
where \( E_{\beta}(\cdot) \) denoting posterior expectation with respect to the posterior density of \( \beta \). By a result of Zellner, the (unique) Bayes estimator of \( \beta \), denoted by \( \hat{\beta}_L \) under the LINEX loss is the value \( \hat{\beta} \) which minimizes (4.23), is given by
\[
\hat{\beta}_L = \frac{1}{c} \ln \left( E_{\beta}[\exp(-c\beta)] \right),
\]
provided that the expectation \( E_{\beta}[\exp(-c\beta)] \) exists and is finite.

4.1 Estimation under Squared Error Loss Function:
4.1.1 Bayes Estimation of the Vector of Parameters:

The two parameters, \( \theta_1 \) and \( \theta_2 \), can be approximately estimated using Lindley’s approximation from (4.4), and their estimates are obtained as follows:

i. The Bayes estimate of the parameter \( \theta_1 \)

Set \( \phi(\theta) = \theta_1 \) in (4.4). Then
\[
\phi = \frac{\partial \phi}{\partial \theta} = 1, \quad \phi = \frac{\partial \phi}{\partial \theta} = 0,
\]
for \( i, j = 1, 2 \),
\[
S = 0, \quad S_{12} = \sigma_{11}, \quad S_{21} = \sigma_{12},
\]
\[
v_{12} = \sigma_{11}^2, \quad v_{21} = \sigma_{21} \sigma_{22},
\]
\[
c_{12} = 3\sigma_{11}^2 \sigma_{12}, \quad c_{21} = \sigma_{22} \sigma_{11} + 2 \sigma_{21}^2.
\]

Substituting the above functions and (4.14-17) in (4.4) yields the Bayes estimate under squared error loss.
function, \(\hat{\theta}_{L,S}\), of \(\theta_1\).

ii. The Bayes estimate of the parameter \(\theta_2\)

Set \(\phi(\theta) = \theta_2\) in (4.4). Then

\[
\phi_1 = \frac{\partial \phi}{\partial \theta_1} = 0, \quad \phi_2 = \frac{\partial \phi}{\partial \theta_2} = 1,
\]

for \(i, j = 1, 2\), \(\phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = 0\),

\(S = 0, \quad S_{12} = \sigma_{21}, \quad S_{21} = \sigma_{22}, \quad v_{12} = \sigma_{12} \sigma_{11}, \quad v_{21} = \sigma_{22}^2, \quad c_{12} = \sigma_{11} \sigma_{22} + 2 \sigma_{12}^2, \quad c_{21} = 3 \sigma_{22} \sigma_{21}\).

Substituting the above functions and (4.14-17) in (4.4) yields the Bayes estimate under squared error loss function, \(\hat{\theta}_{L,S}\), of \(\theta_2\).

4.1.2 The Bayes Estimate of the RF: Set \(\phi(\theta) = R(t)\) in (4.4), where \(R(t)\) is given as in (1.13). Then, for \(j = 1, 2\),

\[
\phi_j = \frac{\partial \phi}{\partial \theta_j} = \frac{\partial R(t)}{\partial \theta_j} = -p_j f_j'(t) \alpha(t), \quad \phi_{12} = \phi_{21} = \frac{\partial^2 \phi}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 R(t)}{\partial \theta_1 \partial \theta_2} = 0
\]

and \(\phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 R(t)}{\partial \theta_i \partial \theta_j} = -p_j f_j(t) \alpha(t)\),

(4.25)

where \(f_j(t), \quad F_j(t)\) and \(\alpha(t)\) are as given, respectively, by (1.1), (1.2) and (3.7).

Substituting (4.25), (4.26) and (4.14-17) in (4.4) yields the Bayes estimate under squared error loss function, \(\hat{R}_{S}\), of \(R(t)\).

4.1.3 The Bayes Estimate of the HF: Set \(\phi(\theta) = H(t)\) in (4.4), where \(H(t)\) is given as in (1.14). Then, for \(j = 1, 2\),

\[
\phi_j = \frac{\partial \phi}{\partial \theta_j} = \frac{\partial H(t)}{\partial \theta_j} = \frac{1}{[R(t)]^2} \left\{ R(t) \frac{\partial f_j(t)}{\partial \theta_j} - f(t) \frac{\partial R(t)}{\partial \theta_j} \right\} = \frac{p_j}{[R(t)]^2} \left\{ R(t) f_j(t) \kappa_j(t) + f(t) F_j(t) \alpha(t) \right\},
\]

(4.27)

\[
\phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 \phi_j}{\partial \theta_i \partial \theta_j} = \frac{E_1 - E_2}{[R(t)]^4},
\]

(4.28)

where
\[ E_1 = [R(t)]^2 \left\{ \frac{\partial R(t)}{\partial \theta_j} \frac{\partial f(t)}{\partial \theta_j} + R(t) \frac{\partial^2 f(t)}{\partial \theta_j^2} - \left[ \frac{\partial f(t)}{\partial \theta_j} \frac{\partial R(t)}{\partial \theta_j} + f(t) \frac{\partial^2 R(t)}{\partial \theta_j^2} \right] \right\} \]

\[ = \rho [R(t)]^2 \{ R(t) f_j(t) [\kappa_j(t) + \kappa_j(t) + \omega(t) f(t) f_j(t)] \} \]

\[ E_2 = 2R(t) \frac{\partial R(t)}{\partial \theta_j} \left\{ R(t) \frac{\partial f(t)}{\partial \theta_j} - f(t) \frac{\partial R(t)}{\partial \theta_j} \right\} \]

\[ = -2R(t) \rho_j^2 (t) F_j(t) \omega(t) \{ R(t) f_j(t) \kappa_j(t) + f(t) F_j(t) \omega(t) \} \],

where \( f_j(t), F_j(t), \kappa_j(t), \kappa_j(t) \) and \( \omega(t) \) are as given, respectively, by (1.1), (1.2), (3.6), (4.21) and (3.7).

For \( i, j = 1, 2 \) and \( i \neq j \),

\[ \phi_{\theta_i} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = \frac{E_1^* - E_2^*}{[R(t)]^4}, \]

where

\[ E_1^* = [R(t)]^2 \left\{ \frac{\partial R(t)}{\partial \theta_i} \frac{\partial f(t)}{\partial \theta_j} - \frac{\partial f(t)}{\partial \theta_i} \frac{\partial R(t)}{\partial \theta_j} \right\} \]

\[ = -[R(t)]^2 \rho_i \rho_j \omega(t) \{ F_i(t) f_j(t) \kappa_j(t) - F_i(t) F_j(t) \kappa_i(t) \} \]

\[ E_2^* = 2R(t) \frac{\partial R(t)}{\partial \theta_i} \left\{ R(t) \frac{\partial f(t)}{\partial \theta_j} - f(t) \frac{\partial R(t)}{\partial \theta_j} \right\} \]

\[ = -2R(t) \rho_i \rho_j F_i(t) \omega(t) \{ R(t) f_i(t) \kappa_i(t) + f(t) F_i(t) \omega(t) \}. \]

Substituting (4.27-29) and (4.14-17) in (4.4) yields the Bayes estimate under squared error loss function, \( \hat{H} \), of \( H(t) \).

### 4.2 Estimation Under LINEX Loss Function

On the basis of the LINEX loss function (4.24), the Bayes estimate of a function \( w = w(\theta_1, \theta_2) \) of the unknown parameters \( \theta_1 \) and \( \theta_2 \) is given by

\[ \hat{w}_L = -\frac{1}{c} \ln E(e^{-cw} | \hat{t}), \quad c \neq 0, \]

where
Suppose \[ \phi(\theta) = e^{-\phi(\theta)} \], so we can apply Lindley's approximation cited previously as it was used to evaluate (4.4). So we obtain the following:

### 4.2.1 Bayes Estimation of the Vector of Parameters

The two parameters, \( \theta_1 \) and \( \theta_2 \) can be approximately estimated using Lindley's approximation from (4.4), and their estimates are obtained as follows:

**i. The Bayes estimate of the parameter \( \theta_1 \)**

Set \( \phi(\theta) = e^{-\phi(\theta)} \) in (4.4). Then

\[
\phi_1 = \frac{\partial \phi}{\partial \theta_1} = -c e^{-\phi_1}, \quad \phi_{11} = \frac{\partial^2 \phi}{\partial \theta_1^2} = c^2 e^{-\phi_1}, \quad \phi_2 = \frac{\partial \phi}{\partial \theta_2} = 0, \quad \phi_{22} = \frac{\partial^2 \phi}{\partial \theta_2^2} = 0,
\]

for \( i, j = 1, 2, \) and \( i \neq j, \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = 0, \)

Substituting the above functions and (4.14-17) in (4.4) then into (4.30) yields the Bayes estimate under LINEX loss function, \( \hat{\theta}_{1,\text{LINEX}} \), of \( \theta_1 \).

**ii. The Bayes estimate of the parameter \( \theta_2 \)**

Set \( \phi(\theta) = e^{-\phi(\theta)} \) in (4.4). Then

\[
\phi_2 = \frac{\partial \phi}{\partial \theta_2} = -c e^{-\phi_2}, \quad \phi_{22} = \frac{\partial^2 \phi}{\partial \theta_2^2} = c^2 e^{-\phi_2}, \quad \phi_1 = \frac{\partial \phi}{\partial \theta_1} = 0, \quad \phi_{11} = \frac{\partial^2 \phi}{\partial \theta_1^2} = 0,
\]

for \( i, j = 1, 2, \) and \( i \neq j, \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = 0, \)

Substituting the above functions and (4.14-17) in (4.4) then into (4.30) yields the Bayes estimate under LINEX loss function, \( \hat{\theta}_{2,\text{LINEX}} \), of \( \theta_2 \).

### 4.2.2 The Bayes Estimate of the RF

Set \( \phi(\theta) = e^{-\phi(\theta)} \) in (4.4), where \( R(t) \) is given as in (1.13). Then, for \( j = 1, 2, \)

\[
\phi_j = \frac{\partial \phi}{\partial \theta_j} = \frac{\partial e^{-\phi(\theta)}}{\partial \theta_j} = c e^{-\phi(\theta)} p_j F_j(t) \alpha(t), \quad (4.32)
\]
for $i, j = 1, 2$, and $i \neq j$, 
\[
\phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = c^2 p_i p_j f_i(t) f_j(t) \omega(t) e^{-cR(t)},
\] (4.33)

and 
\[
\phi_{ii} = \frac{\partial^2 \phi}{\partial \theta_i^2} = \frac{\partial^2 e^{-cR(t)}}{\partial \theta_i^2} = cp_i \omega(t) e^{-cR(t)} \left[ f_j(t) + cp_j \omega(t) F_j^2(t) \right],
\] (4.34)

where $f_j(t), F_j(t)$ and $\omega(t)$ are as given, respectively, by (1.1), (1.2) and (3.7).

Substituting (4.32-34) and (4.14-17) in (4.4) then into (4.30) yields the Bayes estimate under LINEX loss function, $\widehat{R}_L$, of $R(t)$.

4.2.3 The Bayes Estimate of the HF: Set $\phi(\theta) = e^{-cH(t)}$, (4.4), where $H(t)$ is given as in (1.14). Then,

for $j = 1, 2$,
\[
\phi_j = \frac{\partial \phi}{\partial \theta_j} = \frac{\partial e^{-cH(t)}}{\partial \theta_j} = -ce^{-cH(t)} \frac{\partial H(t)}{\partial \theta_j},
\]

where $\frac{\partial H(t)}{\partial \theta_j}$ is defined in (4.27), then
\[
\phi_j = \frac{-cp_j e^{-cH(t)}}{[R(t)]^2} \delta_j,
\] (4.35)

where
\[
\delta_j = R(t)f_j(t)\kappa_j(t) + f(t)F_j(t)\omega(t),
\] (4.36)

\[
\phi_{ij} = \frac{\partial^2 \phi}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 e^{-cH(t)}}{\partial \theta_i \partial \theta_j}
\]
\[
= \frac{-cp_j e^{-cH(t)}}{[R(t)]^4} \left\{ \left[f(t) - \frac{cp_j \delta_j}{[R(t)]^2} \right][R(t)]^2 + 2R(t)F_j(t)\omega(t) \delta_j(t) \right\},
\]

for $i, j = 1, 2$, and $i \neq j$, 
\[
\phi_j = \frac{\partial \phi}{\partial \theta_j} = \frac{-cp_p e^{-cH(t)}}{[R(t)]^4} \left\{ \left[f(t) - \frac{cp_j \delta_j}{[R(t)]^2} \right][R(t)]^2 + 2R(t)F_j(t)\omega(t) \delta_j(t) \right\},
\] (4.38)
where $f_j(t), F_j(t), \kappa_j(t), \kappa_j(t), \delta_j(t)$ and $\omega(t)$ are given, respectively, by (1.1), (1.2), (3.6), (4.21), (4.36) and (3.7).

Substituting (4.35), (4.37), (4.38) and (4.14-17) in (4.4) then into (4.30) yields the Bayes estimate under LINEX loss function, $\hat{H}_L$, of $H(t)$.

5. Simulation Study: We obtained, in the above Sections, Bayesian and non-Bayesian estimates of the vector parameters $\theta_j$, reliability, $R(t)$ and failure rate, $H(t)$ functions of a mixture of two $EG(\theta_j, j=1, 2$, components.

We adopted the squared error loss and LINEX loss functions. The MLE's are also obtained.

In order to assess the statistical performances of these estimates, a simulation study is conducted. The computations are carried out for censoring percentages of 80% and 100% (complete sample case), for each sample size. The mean square errors (MSE’s) using generated random samples of different sizes are computed for each estimator. The random samples are generated as follows:

1. For given values of the prior parameters $\alpha_1$ and $\alpha_2$ generate a random values for $\theta_1$ and $\theta_2$ from the gamma distribution $G(\alpha_j, 1), j=1, 2$.

2. Using $\theta_1$ and $\theta_2$ obtained in step (1), generate random samples of different sizes: $n = 15, 25$ and $50$ from a mixture of two $EG(\theta_j, j=1, 2$, components as given by (1.11). The computations are carried out for such sample sizes and censored samples of sizes: $12, 20$ and $40$, respectively.

3. The MLE’s $\hat{\theta} = (\hat{\theta}_1, M, \hat{\theta}_2, M)$ of the vector parameters $\theta = (\theta_1, \theta_2)$ are obtained by iteratively solving (3.8). The estimators $\hat{R}_M(t_0)$ and $\hat{H}_M(t_0)$ of the functions $R(t)$ and $H(t)$ are then computed at some values $t_0$.

4. The Bayes estimates relative to squared error loss, $\hat{\theta}_2, \hat{R}_2,$ and $\hat{H}_2$ are computed, using (4.4) after considering the appropriate changes according to Subsections (4.1.1), (4.1.2) and (4.1.3). Also, the Bayes estimates relative to LINEX loss $\hat{\theta}_L, \hat{R}_L,$ and $\hat{H}_L$ are computed, using (4.4) after considering the appropriate changes according to Subsections (4.2.1), (4.2.2) and (4.2.3).

5. The above (2-4) steps are repeated 1000 times and the biases and the mean square errors are computed for different sample sizes $n$ and censoring sizes $r$.

The computational (our) results were computed by using Mathematica 4.0. In all above cases the prior parameters chosen as $\alpha_1 = 2$ and $\alpha_2 = 0.5$ which yield the generated values of $\theta_1 = 2.01307$ and $\theta_2 = 0.56372$ (as the true values). The true values of $R(t)$ and $H(t)$ when $t = t_0 = 0.5$, are computed to be $R(0.5) = 0.86723$ and $H(0.5) = 0.312305$. The bias (first entries) and MSE’s (second entries) are displayed in Tables 1-4. The computations are achieved under complete and censored samples.

Tables 1, 2, 3 and 4 contains estimated biases and MSE’s of MLE’s, Bayes estimators under quadratic and LINEX loss functions of $\theta_1, \theta_2, R(t)$ and $H(t)$ respectively.

6. Concluding Remarks

In this paper we have presented the Bayesian and maximum likelihood estimates of the vector parameters $\theta_j$, reliability, $R(t)$ and failure rate, $H(t)$ functions of the lifetimes follow a mixture of two $EG(\theta_j, j=1, 2$, components. The estimation are conducted on the basis of complete and type-II censored samples. Bayes estimators, under squared error loss and LINEX loss functions, are derived in approximate forms by using...
Lindley’s method. The MLE’s are also obtained.

Our observations about the results are stated in the following points:

1. Table 1 shows that the Bayes estimates under the quadratic loss function are the best estimates as compared with the biases of estimates under LINEX loss function or MLE’s. This is True for both complete and censored samples. It is immediate to note that MSE’s decrease as sample size increases. On the other hand the Bayes estimates under the LINEX loss function have the smallest estimated MSE’s as compared with the estimates under quadratic loss function or MLE’s. This is True for both complete and censored samples.

2. Table 2 shows that the Bayes estimates under quadratic loss function have the smallest estimated MSE’s as compared with the Bayes estimates under LINEX loss function or MLE’s. This is true only for complete samples. On the other hand the Bayes estimates under the quadratic loss function have the best bias as compared with the estimates under LINEX loss function or MLE’s for censored samples. Also, we note that MSE’s usually decrease as a complete sample size increases. Further, Bayes estimates under LINEX loss function have the smallest MSE’s. This is True for small and moderate censored samples.

Table 1: Estimated biases (first entries) and MSE’s (second entries) of various estimators of \( \hat{\theta}_1 \), for different sample sizes.

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>( \hat{\theta}_{1,M} )</th>
<th>( \hat{\theta}_{1,S} )</th>
<th>( \hat{\theta}_{1,L}, c=2.5 )</th>
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Table 2: Estimated biases (first entries) and MSE’s (second entries) of various estimators of \( \hat{\theta}_2 \), for different sample sizes.

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<th>n</th>
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<td></td>
<td>0.500746</td>
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<td>0.06269</td>
</tr>
<tr>
<td>50</td>
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<td>0.567017</td>
<td>0.016411</td>
<td>-0.011661</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.499627</td>
<td>0.003708</td>
<td>0.00643</td>
</tr>
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</table>

Table 3: Estimated biases (first entries) and MSE’s (second entries) of various estimators of \( \hat{R}(t) \), for different sample sizes.

<table>
<thead>
<tr>
<th>n</th>
<th>r</th>
<th>( \hat{R}_{1,M}(t) )</th>
<th>( \hat{R}_{2,S}(t) )</th>
<th>( \hat{R}_{1,L}, c=5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>12</td>
<td>-0.135564</td>
<td>0.048574</td>
<td>0.007244</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5.20895</td>
<td>0.048122</td>
<td>0.004998</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>-0.094805</td>
<td>0.061839</td>
<td>0.048557</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.9792</td>
<td>0.044025</td>
<td>0.002935</td>
</tr>
<tr>
<td>25</td>
<td>20</td>
<td>-0.119424</td>
<td>0.040625</td>
<td>-0.005528</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.47587</td>
<td>0.03999</td>
<td>0.003919</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
<td>-0.078305</td>
<td>0.044069</td>
<td>0.03465</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.38539</td>
<td>0.013983</td>
<td>0.00087</td>
</tr>
<tr>
<td>50</td>
<td>40</td>
<td>-0.001362</td>
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<td>0.017442</td>
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<td>0.003671</td>
<td>0.02917</td>
<td>0.003474</td>
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<tr>
<td>50</td>
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<td>-0.018272</td>
<td>0.028585</td>
<td>0.024907</td>
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<tr>
<td></td>
<td></td>
<td>0.15388</td>
<td>0.004704</td>
<td>0.022698</td>
</tr>
</tbody>
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Table 4: Estimated biases (first entries) and MSE's (second entries) of various estimators of \( H(t) \) for different sample sizes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( r )</th>
<th>( \widehat{H}_{ML} (t) )</th>
<th>( \widehat{H}_L(t) )</th>
<th>( \overline{H}_L(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
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<td>0.056278</td>
<td>0.017486</td>
<td>0.098626</td>
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<tr>
<td></td>
<td></td>
<td>0.678701</td>
<td>0.059764</td>
<td>0.0637</td>
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<tr>
<td>15</td>
<td>15</td>
<td>0.085643</td>
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<td>0.046825</td>
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<tr>
<td></td>
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<td>1.128129</td>
<td>0.05686</td>
<td>0.026768</td>
</tr>
<tr>
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<td>20</td>
<td>0.094125</td>
<td>0.033006</td>
<td>0.084909</td>
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<td>0.068893</td>
<td>0.049473</td>
</tr>
<tr>
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<td>25</td>
<td>0.090051</td>
<td>0.027403</td>
<td>0.03003</td>
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<td></td>
<td></td>
<td>0.273923</td>
<td>0.034304</td>
<td>0.018012</td>
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<tr>
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<td>40</td>
<td>0.069277</td>
<td>0.016413</td>
<td>0.063614</td>
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<tr>
<td></td>
<td></td>
<td>0.044136</td>
<td>0.041539</td>
<td>0.034468</td>
</tr>
<tr>
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<td>50</td>
<td>0.112697</td>
<td>0.017439</td>
<td>0.001013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.29472</td>
<td>0.006246</td>
<td>0.002368</td>
</tr>
</tbody>
</table>

3. Table 3 shows that the MLE's have the best bias as compared with the Bayes estimates under LINEX loss function or quadratic loss function for large samples for both complete and censored cases. For small and moderate samples Bayes estimates under the LINEX loss function have the best bias as compared with the estimates under LINEX loss function or MLE's. This is True for both complete and censored samples. It is immediate to note that MSE's usually decrease as sample size increases. On the other hand the Bayes estimates under the LINEX loss function have the smallest estimated MSE's as compared with the estimates under quadratic loss function or MLE's. This is True for both complete and censored samples.

4. Table 4 shows that the Bayes estimates under the LINEX loss function usually have the smallest estimated MSE's as compared with the estimates under quadratic loss function or MLE's. This is True for both complete and censored samples. It is immediate to note that MSE's usually decrease as sample size increases. On the other hand the Bayes estimates under the quadratic loss function usually have the best bias as compared with the estimates under LINEX loss function or MLE's.

From the previous observations, the estimation from a finite mixture of two \( EG \) components data is possible and flexible using Bayes approach, especially using asymmetric loss function such as LINEX function, which is the most appropriate for all parameters as shown from this article. We also, recommend to use Bayes estimates under quadratic loss function for \( \theta \) and \( H(t) \).

REFERENCES


34. Shawky, A.I. and R.A. Bakoban, 2009c. Certain Characterizations of the exponentiated gamma distribution. Accepted for publication in JATA.


