The Application of Cubic Spline Collocation to the Solution of Integral Equations

[1]Oladejo S.O., [1]Mojeed T.A and [2]Olurode K.A.

[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2]Mathematics Department, Obafemi Awolowo University, Ile-Ife,Nigeria.

Abstract: This paper deals with the application of cubic spline collocation to the solution of integral equations. Three cubic spline collocation methods are proposed, namely: standard, chebyshev or orthogonal and equally spaced cubic spline collocation methods. They were applied to some Fredholm integral equations after the integrals have been evaluated. Numerical computations are carried out in order to compare the three methods on the basis of computational costs, efficiency and accuracy. The three techniques produced good numerical solution to the integral equation but comparison of the three methods reveals that chebyshev cubic spline collocation gives the best result with minimum error as the analysis was done under the same numerical experiment.

Key words: collocation methods, Integral equation, spline, cubic spline

INTRODUCTION

Over the years, much emphasis has not been placed on the numerical solution of integral equations, it may be due to complexities or difficulties involve in solving problem associated with it[1,2]. But due to its wide area of application especially in the field of science and engineering, then its solution called for attention. Although, few authors, the likes of Onumanyi and Taiwo[3,4], have only worked on collocation of singularly perturbed second order differential equation while Domingo[5] used collocation tau method in his work. Meanwhile, in this work we consider the application of cubic spline to solution of Fredholm integral equation.

Therefore in this paper, the general integral equation[1,3] used is of the form

$$u(x) + \lambda \int_a^b k(x,s)\, u(s)\, ds = f(x) \tag{1}$$

where

- λ is a scalar parameter, $U(s)$ is known (and often called the driven term),
- $U(x)$ is an unknown function, $f(x)$ is a given function, and
- $k(x,s)$ is the kernel.

The approximate solution[1,3] used is defined as:

$$u(x) = \frac{1}{6h}\left[(x_j - x)^3 m_{j-1} + (x - x_{j-1})^3 m_j\right] + \frac{1}{h}\left[y_{j-1} - \frac{h^2}{6} m_{j-1}\right](x_j - x) +$$

$$\frac{1}{h}\left[y_j - \frac{h^2}{6} m_j\right](x - x_{j-1}) \tag{2}$$

Along with the cubic recurrence relation[1,3]

Corresponding Author: Olurode K.A, Mathematics Department, Obafemi Awolowo University, Ile-Ife,Nigeria.

$$m_{j-1} + 4m_j + m_{j+1} = \frac{6}{h^2}\left[y_{j-1} - 2y_j + y_{j+1}\right] \tag{3}$$

MATERIALS AND METHODS

For the purpose of our discussion, we shall consider a Fredholm integral equation of the second kind of the form[1,3].

$$u(x) + \int_{s_{j-1}}^{s_j} k(x,s)\,u(s)\,ds = f(x), \qquad S_{j-1} \le x \le S_j \;\ldots \tag{4}$$

Thus, substituting equation (2) into equation (4), we obtain,

$$\left\{\frac{1}{6h}\left[x_j^3 - 3x_j^2 x + 3x_j x^2 - x^3\right] - \frac{h}{6}\left[x_j - x\right]\right\}m_{j-1} +$$

$$\left\{\frac{1}{6h}\left[x^3 - 3x^2 x_{j-1} + 3x\,xx_{j-1}^2 - x_{j-1}^3\right] - \frac{h}{6}\left[x - x_{j-1}\right]\right\}m_j +$$

$$\frac{1}{h}\left[x_j - x\right]y_{j-1} + \frac{1}{h}\left[x - x_{j-1}\right]y_j +$$

$$\int_{s_{j-1}}^{s_j} k(x,s)\left\{\begin{array}{l}\left[\frac{1}{6h}\left(s_j^3 - 3s_j^2 s + 3s_j s^2 - s^3\right) - \frac{h}{6}\left(s_j - s\right)\right]m_{j-1} + \\[2mm] \left[\frac{1}{6h}\left(s^3 - 3s^2 s_{j-1} + 3ss_{j-1}^2 - s_{j-1}^3\right) - \frac{h}{6}\left(s - s_{j-1}\right)\right]m_j + \\[2mm] \frac{1}{h}\left[s_j - s\right]y_{j-1} + \frac{1}{h}\left[s - s_{j-1}\right]y_j\end{array}\right\}ds = f(x) \tag{5}$$

In order to simplify equation (5), we let

$$s = s_{j-1} + ph$$

Therefore equation (5) becomes

$$\left\{\frac{1}{6h}\left[x_j^3 - 3x_j^2 x + 3x_j\,x^2 - x^3\right] - \frac{h}{6}\left[x_j - x\right]\right\}m_{j-1} +$$

$$\left\{\frac{1}{6h}\left[x^3 - 3x^2 x_{j-1} + 3xx_{j-1}^2 - x_{j-1}^3\right] - \frac{h}{6}\left[x - x_{j-1}\right]\right\}m_j +$$

$$\frac{1}{h}\left[x_j - x\right]y_{j-1} + \frac{1}{h}\left[x - x_{j-1}\right]y_j +$$

$$\sum_{j=1}^{n} \int_0^1 k(x, s_{j-1} + ph) \left\{ \begin{array}{l} \left[\frac{h^2}{6} \left(1 - 3p^3 + 3p^2 - p\right) - \frac{1}{h} \left(p - 1\right) \right] m_{j-1} + \\ \left[\frac{h^2}{6} \left(p^2 - \frac{1}{h}\right) m_j + \frac{1}{h}(1 - p) \, y_{j-1} + \frac{1}{h} \, py_j \right] \end{array} \right\} dp = f(x) \quad (6)$$

In equation (6), the integrals,

$$\int_0^1 k(x, s_{j-1} + ph) \, p^m \, dp, \qquad m = 0, 1, 2 \text{ and } 3 \tag{7}$$

has to be evaluated. After evaluating the integral, we then collocate the remaining equations at $x = x_i$ using the following three collocation methods.

Standard Collocation Method: In this subsection, we shall consider the standard collocation method. This method requires an equal spacing of collocation points with specified range of the problem at hand[1,2]. If we have an integral equation of the form,

$$y(x) + \lambda \int_a^b k(x, s) \, y \, (s) \, ds = f(x) \tag{8}$$

The problem within the integrand is solved after which we then collocate the entire equation at ($x_k = kh$, $k = 1, 2, \ldots, N+1$ step length),

$$h = \frac{(b-a)}{N}$$

$$x_k = \frac{(b-a)k}{(N+1)}, \quad k = 1, 2, \ldots, N+1$$

where N represents the degree of approximation

This generates a complete algebraic system of equations (along with the recurrence relation and the end conditions), which we then solve simultaneously by Gaussian elimination.

Chebyshev Collocation Method: In this subsection, we shall consider the orthogonal collocation method. The orthogonal collocation method based on the zeros of chebyshev polynomials[1,6,7], by the formula

$$x_i = \frac{1}{2} + \frac{1}{2} \cos \left[\frac{(2i + 1)\pi}{2N} \right]; \quad i = 1, 2, \ldots, N + 2$$

Equally Spaced Collocation Method: In this subsection, we shall consider the equally spaced collocation method. The equally spaced collocation method[8,9] is given by the formula

$$x_i = a + \frac{(b-a)i}{N+2}; \quad i = 1, 2, \ldots, N + 2$$

In each case this leads to a system of algebraic equations along with the recurrence relation given in equation (3).

These algebraic equations are then solved using MATLAB 7.0 to obtain the unknown constants, which are then substituted in the approximate solution.

We shall demonstrate this on some numerical examples and investigate the efficiency, accuracy, and amount of work done in carrying out the three numerical techniques.

Numerical Experiment: We define error in all cases as,

$$Error = Max \left| \underline{Exact - Approx.} \right| = Max \left| y(x) - y_N(x) \right| a \leq X \leq b$$

In the tables below, we present the computational results for the examples considered in this section.

Example 1: Consider the integral equation

$$\int_0^1 e^{xs} y(s)\, ds \; + \; y(x) \;\; = \;\; \frac{e^{x+1} - 1}{(x+1)} \tag{9}$$

Where

$$y(x) = \frac{1}{6h}\left[(x_j - x)^3 m_{j-1} + (x - x_{j-1})^3 m_j \right] + \frac{1}{h}\left[y_{j-1} - \frac{h^2}{6} m_{j-1} \right](x_j - x) +$$

$$\frac{1}{h}\left[y_j - \frac{h^2}{6} m_j \right](x - x_{j-1})$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solution were tabulated below.

Example 2: Consider the integral equation

$$\int_0^1 (x + t)\, u(t)\, dt \;\; = \;\; u(x) \; - \; \frac{3x}{2} \; + \; \frac{5}{6} \tag{10}$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solution were tabulated below.

Example 3: Consider the integral equation

$$\int_0^1 xe^{xt}\, y(t)\, dt \;\; + \;\; y(x) \;\; = \;\; e^x \tag{11}$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solution were tabulated below.

RESULTS AND DISCUSSION

In this paper, the results of the three collocation methods (standard, Chebyshev and equally spaced collocation methods) used in this work were compared with the exact solution. The results of numerical experiment

Table 1: Comparison of the three methods using Error estimation for Example 1 (N=3)

X	Standard	Chebyshev	Equally spaced
0.2	1.2325271×10^{-2}	1.46634×10^{-3}	6.22673×10^{-3}
0.4	1.066305×10^{-2}	6.1063×10^{-3}	7.199962×10^{-3}
0.6	9.913525×10^{-1}	6.64227×10^{-4}	8.341726×10^{-3}
0.8	$9.90721569 \times 10^{-1}$	2.75827×10^{-4}	9.21667×10^{-3}
1.0	3.556017×10^{-3}	1.04139×10^{-4}	1.1339874×10^{-3}

Table 2: Comparison of the three methods using Error estimation for Example1 (N=4)

X	Standard	Chebyshev	Equally spaced
0.2	1.0452×10^{-3}	2.34156×10^{-4}	6.730403×10^{-3}
0.4	1.08163×10^{-4}	1.29306×10^{-4}	8.104992×10^{-3}
0.6	2.76945×10^{-3}	1.69167×10^{-4}	1.0420043×10^{-3}
0.8	7.587642×10^{-2}	1.62478×10^{-4}	1.3675558×10^{-2}
1.0	1.434641×10^{-2}	1.72991×10^{-4}	1.7871536×10^{-2}

Table 3: Comparison of the three methods using Error estimation for Example2 (N=3)

X	Standard	Chebyshev	Equally spaced
0.2	2.193278×10^{-3}	2.4861×10^{-5}	1.716366×10^{-3}
0.4	1.979518×10^{-3}	2.359×10^{-5}	1.668132×10^{-3}
0.6	1.76759×10^{-3}	2.2958×10^{-5}	1.619898×10^{-3}
0.8	1.55199×10^{-2}	2.206×10^{-5}	1.571664×10^{-3}
1.0	3.38239×10^{-4}	2.18538×10^{-5}	1.52343×10^{-3}

Table 4: Comparison of the three methods using Error estimation for Example2 (N=4)

X	Standard	Chebyshev	Equally spaced
0.2	1.0452×10^{-3}	2.9265×10^{-5}	6.28536×10^{-3}
0.4	1.08163×10^{-4}	2.9456×10^{-5}	6.180396×10^{-3}
0.6	2.769451×10^{-3}	2.9819×10^{-5}	5.981386×10^{-3}
0.8	7.57641×10^{-3}	3.0355×10^{-5}	5.68833×10^{-3}
1.0	1.4346409×10^{-2}	3.10615×10^{-6}	5.30723×10^{-3}

Table 5: Comparison of the three methods using Error estimation for Example3 (N=3)

X	Standard	Chebyshev	Equally spaced
0.2	9.56255×10^{-4}	8.38×10^{-7}	5.090401×10^{-3}
0.4	1.48214×10^{-3}	3.1972×10^{-5}	3.285923×10^{-3}
0.6	3.836974×10^{-3}	5.208×10^{-5}	1.496156×10^{-3}
0.8	6.298925×10^{-3}	9.6152×10^{-5}	3.59899×10^{-4}
1.0	8.677317×10^{-3}	1.45719×10^{-4}	$1.02201243 \times 10^{-1}$

Table 6: Comparison of the three methods using Error estimation for Example3 (N=4)

X	Standard	Chebyshev	Equally spaced
0.2	8.10987×10^{-4}	9.372×10^{-6}	2.0387692×10^{-2}
0.4	2.140173×10^{-3}	1.3614×10^{-5}	1.7923518×10^{-2}
0.6	3.470174×10^{-3}	1.9206×10^{-5}	1.6683031×10^{-2}
0.8	4.80099×10^{-3}	2.6149×10^{-5}	1.6903044×10^{-2}
1.0	6.132616×10^{-3}	3.4443×10^{-5}	1.8504619×10^{-2}

investigated as shown in the Tables (1-6) revealed that the solutions obtained by each of the methods are almost the same. However, in terms of accuracy, efficiency and computational cost, Chebyshev method gave the best results among the three proposed methods because of it closeness to exact solution.

Conclusion: An extension of the Collocation algorithm using cubic spline approach for resolving Fredholm integral equation is proposed here. The method is amenable to the usual mathematical analysis. By means of some test problems, the cubic spline collocation method using standard, Chebyshev, and equally spaced collocation methods are shown to be good numerical techniques for solving the Integral equations. However, orthogonal collocation approach was found to give the best result with minimum error.

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