An Iterative Method with Quartic Convergence for Nonlinear Equations
Based on Modified Homotopy Perturbation Method

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Abstract: Modified homotopy perturbation method (HPM) is applied to construct a new iterative method for solving nonlinear algebraic equations, it is shown that the proposed method has fourth-order convergence. Finally, some numerical examples presented to show the efficiency of the theoretical results and for comparing with some other methods.

Key words: Nonlinear equation, Iterative method, Order of convergence

INTRODUCTION

The development of numerical techniques for solving nonlinear algebraic equations is a subject of considerable interest. There are different methods to approximate the solution of nonlinear equation, such as the Newton-Raphson method with quadratic convergence, Steffensen’s method and many papers deal with nonlinear algebraic equations, e.g. Saeed and Aziz\cite{10}, Javidi\cite{7}, Golbabai ana Javidi\cite{4}, Noor and Noor\cite{8}, Noor\cite{9}, Chun\cite{2} and Abbasbandy\cite{1}. Recently, Golbabai ana Javidi\cite{4} modified Newton-Raphson method for solving the non-linear algebraic equation\( f(x)=0 \), by considering terms up to second order in the Taylor series and using Modified HPM, with third order convergence. In this work, iterative method was constructed by considering terms up to third order in Taylor series and the new iterative method proposed by Golbabai ana Javidi\cite{4}, which is fourth-order convergence.

MATERIALS AND METHODS

Consider the Nonlinear Algebraic Equation
\[ f(x) = 0, \quad x \in R \]  \hspace{1cm} (1)

where \( \alpha \) be a root of it and \( f \) is a \( C^3 \) function on an interval containing \( \alpha \). Using third order Taylor series, the nonlinear Eq. (1) can be written as follows:
\[ f(\gamma)+(x-\gamma)f'(\gamma)+\frac{(x-\gamma)^2}{2!}f''(\gamma)+\frac{(x-\gamma)^3}{3!}f'''(\gamma)=0 \]
\hspace{1cm} (2)
where \( \gamma \) is the initial approximation for a zero of Eq. (1).

In a similar manner of Golbabai and Javidi et al.\cite{4}, to construct iterative method, Eq. (2) can be rewritten as follows:
\[ x = c + N(x) \], \hspace{1cm} (3)
where
\[ c = \gamma - \frac{f(\gamma)}{f'(\gamma)} \] \hspace{1cm} (4)
and
\[ N(x) = -\frac{(x-\gamma)^2}{2f'(\gamma)}f''(\gamma) - \frac{(x-\gamma)^3}{3!f'(\gamma)}f'''(\gamma) \] \hspace{1cm} (5)

To complete the derivation of the iterative method, we use the basic ideas of modified homotopy perturbation method, we construct a homotopy\( \Theta : (R \times [0,1]) \times R \to R \) for Eq. (3) which satisfies
\[ \Theta(\sigma, p, \theta) = \sigma - c - pN(\sigma) \]
\[ + p(1-p)\theta = 0, \sigma, \sigma \in R, p \in [0,1] \] \hspace{1cm} (6)
where \( \theta \) is an unknown real number and \( p \) is embedding parameter. See\cite{9,10}

It is obvious that
\[ \Theta(\sigma, 0, \theta) = \sigma - c = 0 \] \hspace{1cm} (7)
\[ \Theta(\sigma, 1, \theta) = \sigma - c - N(\sigma) = 0. \] \hspace{1cm} (8)

Suppose a solution of Eq. (3) having the series form:
\[ \sigma = \sum_{i=0}^{\infty} x_i p^i \] \hspace{1cm} (9)
Which is approximate solution of Eq. (1), therefore, can be readily obtained:

$$\alpha = \lim_{p \to 1} \sigma = \sum_{i=0}^{\infty} x_i$$  \hspace{1cm} (10)

For the application of modified HPM to (1), we can write (6) as follows: by expanding $N(\sigma)$ into Taylor series around $x_0$

$$\sigma - c - p \left\{ \frac{N(x_0) + (\sigma - x_0)}{1!} \cdot \frac{N'(x_0)}{2!} \cdot \sum_{n=0}^{\infty} \frac{N''(x_0)}{2!} \right\} + p(1 - p) \theta = 0$$  \hspace{1cm} (11)

Substitution of (9) into (11) yields

$$x_0 + p^0 x_1 + p^2 x_2 + \cdots - c - p \left\{ N(x_0) + (x_0 + p x_1 + p^2 x_2 + \cdots - x_0) \cdot \frac{N'(x_0)}{1!} \right. \\
+ (x_0 + p x_1 + p^2 x_2) + \cdots - x_0)^2 \cdot \frac{N''(x_0)}{2!} + \cdots \right\} - p(1 - p) \theta = 0$$  \hspace{1cm} (12)

By equating the terms with identical powers of $p$, we have

$$p^0 : x_0 - c = 0,$$  \hspace{1cm} (13)

$$p^1 = x_1 - N(x_0) - \theta = 0,$$  \hspace{1cm} (14)

$$p^2 : x_2 - x_1 N'(x_0) + \theta = 0$$  \hspace{1cm} (15)

$$p^3 : x_3 - x_2 N'(x_0) + \frac{1}{2} x_2^2 N''(x_0) = 0$$  \hspace{1cm} (16)

We try to find parameter $\theta$, such that

$$x_2 = 0$$  \hspace{1cm} (17)

Hence by substituting $x_i = N(x_0) + \theta$ from (14) into (15) and using (17), we have

$$\theta = \frac{N(x_0) N'(x_0)}{1 - N'(x_0)}$$  \hspace{1cm} (18)

By substituting (18) into (14), we have

$$x_1 = \frac{N(x_0)}{1 - N'(x_0)}$$  \hspace{1cm} (19)

Also, substituting (19) and (17) into (16), we have

$$x_3 = -\frac{1}{2} \left( \frac{N(x_0)}{1 - N'(x_0)} \right)^2 N''(x_0)$$  \hspace{1cm} (20)

Now by substituting (20), (19) and (17) into (10), we can obtain the zero of Eq. (1) as follows

$$\alpha = x_0 + x_1 + x_2 + x_3 + \cdots = c + \frac{N(x_0)}{1 - N'(x_0)}$$

$$= \frac{1}{2} \left( \frac{N(x_0)}{1 - N'(x_0)} \right)^2 N''(x_0) + \cdots$$

This formulation allows us to suggest the following iterative method for solving nonlinear Eq. (1).

**Algorithm AL:** For a given $x_0$, compute the approximate solution $x_{n+1}$ by the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{\int f''(x_n) f'(x_n) \cdot f(x_n) \cdot f''(x_n) - 3 f'(x_n) f''(x_n) \cdot f'(x_n)}{3(2 f'(x_n)^2 - 2 f''(x_n) f'(x_n) + f'(x_n) f''(x_n))}$$  \hspace{1cm} (22)
We consider the convergence of algorithm AL.

**Definition:** Let $e_n = x_n - r$ be the truncation error in the $n$th iterate. If there exists a number $k \geq 1$ and a constant $c \neq 0$ such that

$$
\lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n} \right|^k = c
$$

(23)

then $k$ is called the order of convergence of the method.

**Theorem:** Consider the nonlinear equation $f(x) = 0$. Suppose $f$ is sufficiently differentiable. Then for the iterative method defined by Eq. (22), the convergence is at least of order 4.

**Proof:** Let $r$ be a simple zero of $f$. Since $f$ is sufficiently differentiable, by expanding $f(x_n), f'(x_n), f''(x_n)$ and $f'''(x_n)$ about $r$, we get

$$
f'(x_n) = f'(r)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + \cdots \right]
$$

(24)

$$
f''(x_n) = f'(r)\left[2c_2 + 6c_3 e_n + 12c_4 e_n^2 + 20c_5 e_n^3 + 30c_6 e_n^4 + 42c_7 e_n^5 + \cdots \right]
$$

(25)

$$
f'''(x_n) = f'(r)\left[6c_2 + 24c_3 e_n + 12c_4 e_n^2 + 60c_5 e_n^3 + 120c_6 e_n^4 + 336c_8 e_n^5 + \cdots \right]
$$

(26)

where $c_n = \frac{f^{(n)}(r)}{n! f'(r)}$, $n = 1, 2, 3, \ldots$ and $e_n = x_n - r$.

We can rewrite (22) as follows

$$
e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} + \frac{f^2(x_n) \left[f'''(x_n) f'(x_n) - 3 f''(x_n) f''(x_n) \right]}{3 \left(2 f'(x_n)^2 - 2 f''(x_n) f'(x_n)^2 \right)}
$$

(27)

Now from (24), (25), (26) and (27), we have

$$
e_n - \frac{f(x_n)}{f'(x_n)} = c_2 e_n^2 + (2c_3 - 2c_2^2) e_n^3
+ (4c_3^3 + 3c_4 - 7c_2c_3) e_n^4
+ (20c_2^2c_3 - 8c_4^4 - 10c_3c_4 - 6c_3^2 + 4c_5) e_n^5 + \cdots
$$

(28)

$$
f''(x_n) \left[f'''(x_n) f'(x_n) - 3 f''(x_n) f''(x_n) \right]
= - c_2 e_n^2 + (2c_2^2 - 2c_3) e_n^3 + (-2c_2^3 - 2c_4
+ 7c_2c_3) e_n^4 + (-16c_2^2c_3 + 4c_4^2 + 10c_3c_4 - 6c_3^2) e_n^5 + \cdots
$$

(29)

By adding (28) and (29) we get

$$
e_{n+1} = (c_2^2 + c_4) e_n^4 + (4c_3 c_2^2 - 4c_4^2 + 4c_5) e_n^5 + \cdots
$$

This shows that the method (i.e. Algorithm AL) has fourth-order convergence.

**Numerical Examples:** Five examples are presented here to illustrate the efficiency of the newly developed method in this paper by comparing the method of Noor[9] (NMN), The method Javidi (MJ)[3,4], the methods of Golbabi and Javidi[7] (GJ1) (GJ2), method of Saeed and Aziz[10] (SA), and the Algorithm AL, introduced in the present paper.

The following stopping criterion is used for computer programs: $|f(x_n)| < \varepsilon$, for $\varepsilon = 10^{-10}$

**Note:** $n$ means number of iterations.

$$f_1(x) = \sin^2(x) - x^2 + 1$$

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\[ f_2(x) = x^2 - (1-x)^3 \]

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\[ f_3(x) = x^2 - e^{x^2} - 3x + 2 \]

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\[ f_4(x) = \log(x+10) e^{3x} \sin(x) \]

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\[ f_5(x) = e^x + 2^{-x} + 2 \cos(x) - 6 \]

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**Conclusion:** In this paper, we conclude that the iterative method introduced in this paper (Algorithm AL) performs better than the Newton-Raphson method for solving nonlinear algebraic equation \( f(x) = 0 \), also better than the method proposed by Golbabai and Javidi \(^4\) in which is name “A third order Newton type method for nonlinear equations based on modified homotopy perturbation method” (and some times better than the Saeed and Aziz\(^10\)).

**REFERENCES**
