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Homotopy Perturbation Method for Solving System of Nonlinear Fredholm Integral Equations of the Second Kind

Rostam K. Saeed

Salahaddin University/Erbil - College of Science,
Department of Mathematics, Erbil, Kurdistan Region, Iraq.

Abstract: In this article, the homotopy perturbation method is applied to solve system of nonlinear Fredholm integral equations of the second kind, and a new computer program by using MATLAB 7.0 was established. This program is very useful for solving (m x m; m $\geq$2) a system of nonlinear Fredholm integral equations. Results of the examples indicated that the method and the computer program are effective and simple.

Key words: Systems of Fredholm integral equations; homotopy perturbation method

INTRODUCTION

The homotopy perturbation method (HPM) was proposed first by He[8,9] which is in fact, a coupling of the traditional perturbation method and homotopy in topology. In this method the solution is considered as the sum of an infinite series which is very rapid convergence to the accurate solution.

The application of the HPM in nonlinear problems and many other subjects has been intensively studied by scientists and engineers[4, 9-14], especially in integral equations[1,2,7,15], because this method deforms the difficult problem under study into a simple problem which is easy to solve.

Recently Javidi and Golbabai[15] applied HPM for solving system of linear Fredholm integral equations with difference kernel. In this paper, we extend the HPM for solving system of nonlinear Fredholm integral equations of the second kind, also a general computer program is written for solving this system.

MATERIALS AND METHODS

Consider the following system of nonlinear Fredholm integral equations of the second kind:

$$f_i(x) = g_i(x) + \int_a^b k_i(x,t,f_1(t),f_2(t),\ldots,f_m(t))\,dt, \text{ for } i=1,2,\ldots\ldots,m, \tag{1}$$

where the functions $g_i(x)$ and the kernels $k_i(x,t,f_1(t),f_2(t),\ldots,f_m(t))$ are given, and $f_i(x)$ the solution to be determined for i=1, 2,..., m. We suppose that the system (1) has a unique solution.

Hereunder, we construct homotopy perturbation technique for solving equation (1) as follows: By the homotopy technique given by[4,5,6,8-11], we construct a homotopy for (1) which satisfies:

$$H_i(F,p) = (1-p)G_i(F) + pL_i(F) = 0 \tag{2}$$

$$H_i(F,0) = G_i(F), H_i(F,1) = L_i(F), \tag{3}$$

where

Corresponding Author: Rostam K. Saeed, Salahaddin University/Erbil - College of Science – Department of Mathematics, Erbil, Kurdistan Region, Iraq.
E-mail: rostamkarim64@uni-sci.org

$$G_i(F) = f_i - g_i,\ L_i(F) = f_i(x) - g_i(x) - \int_a^b k_i(x,t,f_1(t),\ldots,f_n(t))\,dt \tag{4}$$

$F(x) = [f_1(x), f_2(x), \ldots, f_n(x)]$ and $p \in [0,1]$ is an embedding parameter.

We assume that the solution of (1) can be expressed in a series of p as follows:

$$f_i(x) \approx f_{i0}(x) + p f_{i1}(x) + p^2 f_{i2}(x) + \ldots p^n f_{in}(x), n \geq 2 \tag{5}$$

as $n \to \infty$ and $p \to 1$ then the series is approach to the exact solution.
Substituting equation (5) into (2), we get

$$H_i(F,p) = (1-p)(f_{i0} + pf_{i1} + p^2 f_{i2} + \cdots + p^n f_{in} - g_i)$$
$$+p[f_{i0} + pf_{i1} + \cdots + p^n f_{in} - g_i - \int_a^b k_i(x,t, f_{10} + pf_{11} + \cdots + p^n f_{1n},$$
$$f_{20} + pf_{21} + \cdots p^n f_{2n}, \cdots, f_{m0} + pf_{m1} + \cdots p^n f_{mn})\,dt] = 0 \tag{6}$$

Suppose

$$h_i(x,p,n) = f_{i0}(x) + pf_{i1}(x) + p^2 f_{i2}(x) + \cdots + p^n f_{in}(x)$$

Then rewrite equation (6) as follows

$$H_i(F,P) = h_i(x,p,n) - g_i(x) - p\int_a^b k_i(x,t,h_1(t,p,n),\cdots,h_m(t,p,n))\,dt = 0 \tag{7}$$

Equating coefficients of like powers of p in (7) yields

$$p^0 : f_{i0} = g_i,$$
$$p^1 : f_{i1} = -\int_a^b k_i(x,t,h_1(t,1,0),h_2(t,1,0),\cdots,h_m(t,1,0))\,dt$$
$$p^2 : f_{i2} = -\int_a^b k_i(x,t,h_1(t,1,1) - h_1(t,1,0), h_2(t,1,1) - h_2(t,1,0),\cdots, h_m(t,1,1) - h_m(t,1,0))\,dt, \tag{8}$$

$$P^n : f_{in} = \int_a^b k_i(x,t,h_1(t,l,n-1) - h_1(t,l,n-2), h_2(t,l,n-1) - h_2(t,l,n-2),\ldots,h_m(t,l,n-1) - h_m(t,l,n-2))\,dt.$$

Using the above equations, we assume that the approximate solution of (1) is given by

$$f_i^N(x) = \sum_{n=0}^{N} f_{in} \text{ for } i=1, 2, \ldots\ldots, m \tag{9}$$

Computer Program: In this section the whole process of the homotopy perturbation method was programmed in MATLAB7.0. Now if we want to solve a system of nonlinear Fredholm integral equations of the second kind (1) based on homotopy perturbation method, everything we have to do is just to input the information about the system, then the program will give out the solution of the system.
In the program, the parameters are set as follows:

a: lower limit of the integrals,
b: upper limit of the integrals,
N: number of expansion terms in equation (9),
K: represents the kernels of the system,
G: represents the given functions of the system.

Note: To input the values of K and G in the program, see the solution of the numerical examples in section 4. Hereunder, a general program for solving m×m (m≥2) system of nonlinear Fredholm integral equations of the second kind based on homotopy perturbation method is given:

```
function ss = Homotopymethod (a,b,N,Ker,G)
syms  x t p fg df  ds1 dd
clc
tic
for I = 1:length (G)
eval (['g' int2str (i), '= char (G(i))' ';']);
d =eval (['g' int2str (i)]);
eval (['ff' int2str (i), '=d' ';']);
end
for j = 1:N-1
 for i=1:length (G)
ds1=eval (['ff' int2str (i)]);
df = ['f' int2str (i) int2str (j)]*p^j;
eval (['ff' int2str (i), '=ds1+df' ';']);
end
for I = 1: length (G)
kkerr = char (Ker (i));
for ii = 1: length (G)
     ds2 = eval (['ff' int2str (ii)]);
     eval (['ff' int2str (ii) 't','=subs (ds2,x,t)' ';']);
     ds4 = eval (['ff' int2str (ii) 't']);
     ds3 = ['f' int2str (ii) 't'];
     kkerr = subs (kkerr,ds3,ds4);
      end
      ds1 = eval (['ff' int2str(i)]);
      gg = eval (['g' int2str(i)]);
      Homotopyequation = collect ((ds1-gg)-p*int (kkerr,t,a,b),p);
      taylorhom1 = taylor (Homotopyequation,j+1,p)-taylor (Homotopyequation,j,p);
      H11 = subs (taylorhom1,{['f' int2str(i) int2str (j)],p},{0,1});
     eval (['f',int2str (i) int2str (j), '= -H11']);
      ds1 = eval (['ff' int2str (i)]);
      ds2 = ['f' int2str (i) int2str (j)];
      ds3 = eval (['f' int2str (i) int2str (j)]);
```

```
    eval (['ff' int2str (i) ' = collect (subs(ds1,ds2,ds3,x))' ';']);
end
end
p=1;
for i=1:length (G)
    D = subs (eval(['ff' int2str (i)]),p,1);
    fprintf ('The approximate solution of %s is',['f' int2str (i) '(x)'])
vpa (eval(D),10)
end
toc
```

RESULTS AND DISCUSSION

To demonstrate the effectiveness of the proposed method, three systems of nonlinear Fredholm integral equations of the second kind was considered. Also, to show the efficiency of the method for solving system of nonlinear Fredholm integral equations of the second kind in comparison with exact solution, we report the absolute error which defined by

$$\left\| f_i^N(x) \right\| = \left| f_i(x) - f_i^N(x) \right|; \text{ i=1, 2,.........,} m$$

where $f_i^N(x)$ given in the equation (8) and $f_i(x)$ is the exact solution of the system (1).

Note: To solve the examples by using a computer program which is given in section 3, save a program as an M-file under the name "Homotopymethod".

Example 1:[3] Consider the following system of nonlinear Fredholm integral equations with the exact solution $f_1(x) = x$ and $f_2(x) = x^2$

$$f_1(x) = x - \frac{5}{18} + \int_0^1 \frac{1}{3}(f_1(t) + f_2(t))dt$$

$$f_2(x) = x^2 - \frac{2}{9} + \int_0^1 \frac{1}{3}(f_1^2(t) + f_2(t))dt$$

Solution: In MATLAB7.0, one just needs to input the following command:
>>Homotopymethod(0,1,10,{'(1/3)*(f1t+f2t)''(1/3)*(f1t^2+f2t)'},{'x-(5/18)' 'x^2-(2/9)'})

We can get the following results:

$$\begin{aligned} f_{10}(x) &= x - \frac{5}{18} & f_{20}(x) &= x^2 - \frac{2}{9} \\ f_{11} &= \frac{1}{9} & f_{21} &= \frac{79}{972} \\ f_{12} &= \frac{187}{2916} & f_{22} &= \frac{127}{2916} \\ f_{19} &= \frac{53}{16923} & f_{29} &= \frac{115}{31529} \end{aligned} \tag{10}$$

Substituting (10) into $f_i^2(x) = \sum_{n=0}^{2} f_{in}$ and $f_i^9(x) = \sum_{n=0}^{9} f_{in}$ for i=1, 2, we get

$f_1^2(x) = x$-.6664e-1 $\qquad f_2^2(x) = x^2$-.9739e-1

$f_1^9(x) = x$-.6926e-1 $\qquad f_2^9(x) = x^2$-.1088e-1

The numerical results obtained for Example 1 are shown in Table 1.

Table 1: Numerical results for Example 1

x	$\|f_i^n(x)\|$			
	$\|f_1^2(x)\|$	$\|f_2^2(x)\|$	$\|f_1^9(x)\|$	$\|f_2^9(x)\|$
0	.6664e-1	.6926e-1	.8830e-2	.1088e-1
0.2	.6664e-1	.6926e-1	.8830e-2	.1088e-1
0.4	.6664e-1	.6926e-1	.8830e-2	.1088e-1
0.6	.6664e-1	.6926e-1	.8830e-2	.1088e-1
0.8	.6664e-1	.6926e-1	.8830e-2	.1088e-1
1	.6664e-1	.6926e-1	.8830e-2	.1088e-1

Example 2: Consider the following system of nonlinear fredholm integral equations

$$f_1(x) = -\frac{10}{9} + \frac{11}{12}x + \int_0^1 (xtf_1^2(t) + t^2 f_2^3(t))\,dt$$

$$f_2(x) = -\frac{1}{252} - \frac{1}{70}x + \frac{4}{5}x^2 + \int_0^1 (tf_1^3(t) - xf_2(t))^2\,dt$$

with the exact solution $f_1(x) = x - 1$ and $f_2(x) = x^2$.

Solution: In MATLAB7.0, input the following command:

```
>> Homotopymethod(0,1,10,{'x*t*f1t^2+t^2*f2t^3' '(t*f1t^3-x*f2t)^2'},{'11/12*x-10/9' '4/5*x^2-1/252-1/70*x'})
```

The results are:

$$f_{10}(x) = -\frac{10}{9} + \frac{11}{12}x \qquad f_{20}(x) = -\frac{1}{252} - \frac{1}{70}x + \frac{4}{5}x^2$$

$$f_{11} = \frac{841}{16021} + \frac{228}{1537}x \qquad f_{21} = \frac{62}{4663} + \frac{224}{7183}x + \frac{202}{1679}x^2$$

$$f_{12} = \frac{179}{5135} - \frac{219}{3214}x \qquad f_{22} = -\frac{110}{9093} - \frac{129}{6971}x + \frac{203}{3614}x^2 \tag{11}$$

$$\vdots$$

$$f_{19} = \frac{1}{370715} + \frac{20}{62459}x \qquad f_{29} = \frac{15}{40262} - \frac{7}{58839}x + \frac{23}{180400}x^2$$

Substituting (11) into $f_i^2(x) = \sum_{n=0}^{2} f_{in}$ and $f_i^9(x) = \sum_{n=0}^{9} f_{in}$ for i=1, 2, we get

$$f_1^2(x) = 1.0108x\text{-}1.0120 \qquad f_1^9(x) = 1.0001x\text{-}1.0002$$

$$f_2^2(x) = .98732x^2 + .2816\text{e-}2x + .2301\text{e-}2 \qquad f_2^9(x) = .9997\,x^2 - .4507\text{e-}4x + .1408\text{e-}3$$

The numerical results obtained for Example 2 are shown in Table 2.

Table 2: Numerical results for Example 2

x	$\lVert f_i^n(x) \rVert$			
	$\lVert f_1^2(x) \rVert$	$\lVert f_2^2(x) \rVert$	$\lVert f_1^9(x) \rVert$	$\lVert f_2^9(x) \rVert$
0	.1200e-1	.2301e-2	.2386e-3	.1408e-3
0.2	.9849e-2	.2357e-2	.2050e-3	.1232e-3
0.4	.7699e-2	.1398e-2	.1714e-3	.8830e-4
0.6	.5548e-2	.5740e-3	.1378e-3	.3619e-4
0.8	.3398e-2	.3561e-2	.1043e-3	.3315e-4
1	.1247e-2	.7561e-2	.7068e-4	.1197e-3

Example 3: Consider the following system of nonlinear fredholm integral equations

$$f_1(x) = \frac{x}{2} - \frac{1}{4} + \int_0^1 [x\, f_1(t) + t\, f_2(t)]\, dt$$

$$f_2(x) = x^2 - \frac{2}{9} + \int_0^1 \frac{1}{3}(f_3^2(t) + f_2(t))\, dt$$

$$f_3(x) = x - \frac{2}{15} + \int_0^1 \frac{1}{5}(f_1(t)\, f_3(t) + f_2(t))\, dt$$

with the exact solution $f_1(x) = x, f_2(x) = x^2 \; and \; f_3(x) = x$.

Solution: In MATLAB7.0, input the following command:

```
>>Homotopymethod(0,1,10,{'x*f1t+t*f2t' '(1/3)*(f3t^2+f2t)' '(1/5)*(f1t*f3t+f2t)'}, 'x/2-(1/4)' 'x^2-(2/9)' 'x-(2/15)'})
```

The results are:

$$f_{10}(x) = \frac{x}{2} - \frac{1}{4} \qquad f_{20}(x) = x^2 - \frac{2}{9} \qquad f_{30}(x) = x - \frac{2}{15}$$

$$f_{11}=\frac{5}{36} \qquad f_{21}=\frac{74}{675} \qquad f_{31}=\frac{11}{360}$$

$$f_{12}=\frac{37}{675}+\frac{5}{36}x \qquad f_{22}=\frac{713}{16200} \qquad f_{32}=\frac{141}{4391} \qquad (12)$$

$$\vdots$$

$$f_{19}=\frac{4}{47953}+\frac{200}{22247}x \qquad f_{29}=\frac{75}{31613} \qquad f_{39}=\frac{76}{30231}$$

Substituting (12) into $f_i^2(x)=\sum_{n=0}^{2} f_{in}$ and $f_i^9(x)=\sum_{n=0}^{9} f_{in}$ for i=1, 2, 3 we get

$$f_1^2(x)=.7631x-.3429e\text{-}1 \qquad f_1^9(x)=.9779x-.4940e\text{-}2$$

$$f_2^2(x)=x^2-.4575e\text{-}1 \qquad f_2^9(x)=x^2-.7507e\text{-}2$$

$$f_3^2(x)=x-.4959e\text{-}1 \qquad f_3^9(x)=x-.7480e\text{-}2$$

The numerical results obtained for Example 3 are shown in Table 3.

Table 3: Numerical results for Example 3

x	$\|f_i^n(x)\|$					
	$\|f_1^2(x)\|$	$\|f_2^2(x)\|$	$\|f_3^2(x)\|$	$\|f_1^9(x)\|$	$\|f_2^9(x)\|$	$\|f_3^9(x)\|$
0	.3429e-1	.4575e-1	.4959e-1	.4940e-2	.7508e-2	.7481e-2
0.2	.8166e-1	.4575e-1	.4959e-1	.9355e-2	.7508e-2	.7481e-2
0.4	.1290	.4575e-1	.4959e-1	.1377e-1	.7508e-2	.7481e-2
0.6	.1764	.4575e-1	.4959e-1	.1818e-1	.7508e-2	.7481e-2
0.8	.2238	.4575e-1	.4959e-1	.2260e-1	.7508e-2	.7481e-2
1	.2711	.4575e-1	.4959e-1	.2701e-1	.7508e-2	.7481e-2

Conclusion: As it can be seen in all examples presented in this paper, to obtain a good approximation to the solution, a large iteration should be done. The results of the method in this paper are much closed to the result of Adomian Decomposition method (see Example 2 of[3]). Also, the results of the examples indicate that the given program of homotopy perturbation method is simple and effective.

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