Sensitivity Analysis of Transportation Problems

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Abstract: This paper presents a sensitivity analysis to multiobjective transportation problem. The proposed approach yields to maximum tolerance percentages in multiobjective transportation problem. The weighted sum approach is used to solve the multiobjective transportation problem. The proposed approach allows changing in both the weights and the objective function coefficients and the right hand side constraints simultaneously and independently from their specified values while remaining the same basis optimal. Formulation of both perturbed solution and the corresponding perturbed objective values are presented. An illustrative example is presented to clarify the idea of the proposed approach.

Key words: Sensitivity analysis; Parametric programming; transportation problem; Tolerance approach.

INTRODUCTION

Sensitivity analysis is to analyze the effect of the changes of the objective function coefficients and the effect of changes of the right hand side constraints on the optimal value of the objective function as well as the validity ranges of these effects. Studies on the sensitivity analysis in linear programming and multiobjective programming had been developed\(^\text{[1]}\text{=}^{\text{[4]}}\). The implementation of sensitivity analysis is based primarily on basic optimal solutions. Degeneracy is a complex phenomenon that may influence the computation of optimal basic solutions and sensitivity analysis. Therefore, it may be difficult to determine the ranges of parameter values for which a given optimal solution remains optimal for the transportation problem. Thus, it has been pointed out that when degeneracy occurs, the sensitivity analysis approach to determine sensitivity ranges of parameters yields impractical results. Apparently traditional sensitivity analysis is not suitable for transportation problem. Saaty and Gass\(^\text{[7]}\) worked on perturbations in the objective function coefficients and in the right hand side terms of the constraints. They expressed these coefficients as linear functions of parameters. Gal\(^\text{[8]}\) had been expressed these coefficients in matrix form. Wendell\(^\text{[9]}\) had presented tolerance approach to sensitivity analysis in linear programming. His approach had considered changing in the objective function coefficients and in the right hand terms of constraints simultaneously and independently. The objective function coefficients and the right hand terms of constraints were changing within the maximum tolerance percentage of their specified values simultaneously and independently while maintaining same optimal solution. Barnett\(^\text{[10]}\) worked on perturbations in the matrix coefficients. He described an algorithm for finding a rectangular region within which the components of a column vector may vary. Shetty\(^\text{[11]}\) had studied the case where a single element of the coefficient matrix was perturbed. The usual analysis of this case, yielded the largest interval in which the same basis was optimal. In general, this interval may not be closed. Saaty and Kim\(^\text{[12,13]}\) had considered perturbations in a column with a scalar parameter. In fact, the analysis of a column perturbation with a scalar parameter can be reduced to the analysis of an element of the coefficient matrix with a scalar parameter. In contrast, the tolerance approach considers a vector perturbation of the column which cannot be further reduced. Ravi and Wendell\(^\text{[14]}\) had presented tolerance approach to sensitivity analysis of matrix coefficients in linear programming. Their approach had considered changing in the elements of a column or a row of the matrix coefficients simultaneously and independently in a standard linear programming problem. Their approach yielded to a maximum tolerance percentage within which the elements of a column may all vary simultaneously and independently from their estimated values while still retaining the same set of basic variables in an optimal solution. Hansen, Labbe and Wendell\(^\text{[15]}\) had presented perturbations in multiple linear programming. Their approach determined the maximum percentage by which all weights can deviate simultaneously and independently from their estimated values while retaining the same optimal basis. They also shown how a priori bounds on the ranges of the weights can be exploited to yield a larger maximum tolerance percentage. Marmol and Puerto\(^\text{[16]}\) had presented special cases of the tolerance approach in multiobjective linear programming. They enlarged the range of meaningful regions of weights that can be handled easily from the tolerance approach point of view. They represented the information that the decision maker is able to offer the relationships between the importance of different objectives, in terms of marginal substitution rates. They also had presented an algorithm which reduced the computational effort, in order to
get the maximum tolerance percentage. Lin and Wen\[17\]
had presented two other types of sensitivity range for
the assignment problem. The first type of sensitivity
range has used to determine the range in which the
current optimal assignment remains optimal. The
second type of sensitivity range was to determine those
values of assignment model parameters for which the
rate of change of optimal value function remains
constant.

In this paper, the tolerance approach to sensitivity
analysis in multiobjective transportation problems is
presented. The weighted sum approach is used to
convert the multiobjective transportation problem into
a single transportation problem. The proposed approach
allows changing in both the weights, the objective
function coefficients and in the right hand side
constraints terms simultaneously and independently
from their specified values while remaining the same
solution optimal. The proposed approach can be
considered as an extension of Wendell\[16\] tolerance
approach in a special case of linear programming and
in the case of multiobjective transportation problem.
Also, the proposed approach can be considered as
another tolerance approach of both Hansen, Labbe and
Wendell\[15\] tolerance approach in a special case of the
transportation problem that allows changing in both the
weights, the objective functions coefficients and in the
right hand side terms simultaneously and independently
from their specified values while remaining the same
solution optimal. The proposed approach differs from
their approach in allowing both weighting and objective
functions coefficients to change parametrically
according to maximum certain specified tolerance
intervals while remaining the same solution optimal. The
weights and coefficients of each objective function
are changed parametrically according to certain
specified intervals on the equivalent objective function
coefficients from the ordinary analysis only to
simultaneous and independent perturbations of the
coefficients of nonbasic variables. Similarly, the
intervals for the right hand side terms are applied only
to simultaneous and independent perturbations of the
terms in those constraints that having basic variables in
the optimal tableau. The condition of balanced
transportation problem is taken into consideration
when dealing with the right hand side constraints. The
proposed tolerance approach uses only the data of
initial and final simplex tableau of the equivalent single
linear problem. Formulation of both obtained perturbed
solution and the corresponding perturbed objective
values are presented. An illustrative example is
presented to clarify the idea of the proposed approach.

**Problem Formulation:** It is well known that the
transportation problem TP is a special case of linear
programming (LP). Consider the multiobjective
transportation problem P(MTP) of K objectives,
each of m sources and n destinations as follows:

$$\begin{align*}
P(MTP): & \text{ Min } \left[ \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}^{(k)} x_{ij}, \quad k=1,2,\ldots, K \right] \\
\text{subject to:} \\
& \sum_{j=1}^{n} x_{ij} = a_i, \quad i=1,2,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j=1,2,\ldots,n \\
& \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad (\text{Balanced condition}) \\
x_{ij} \geq 0, \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n
\end{align*}$$

where,

- $c_{ij}^{(k)}$ is the cost of transporting a unit of objective
- $k$ from source $i$ to destination $j$.
- $x_{ij}$ is the amount transported from source $i$ to
destination $j$.
- $a_i$, $i=1,2,\ldots,m$ are the availability at $i$ source and
- $b_j$, $j=1,2,\ldots,n$ are the requirement at $j$ destination.

The $m$ constraints (2) represent the supplies and
the $n$ constraints (3) represent the destinations. The
balanced condition (4) will lead to reduce the total
number of constraints given by either (2) or (3) by one
i.e., total number of constraints = $m + n - 1$. This
means that the number of optimal basic variables in the
final optimal simples tableau is equal to $m + n - 1$.

The previous multiobjective transportation problem
P(MTP) given by (1-4) can be put in matrix form as
follows:

$$\begin{align*}
P(MTP): & \text{ Min } \left[ C^{(k)} X, \quad k=1,2,\ldots,K \right] \\
\text{subject to:} \\
& A X = B, \quad X \geq 0
\end{align*}$$

where,

- $C^{(k)}$ is a matrix of $m \times n$ dimensional represents
the $K$ objective coefficients,
- $X$ is a column vector of $n$ dimensional represents
the decision variables,
- $A$ is an $(m+n-1) \times (m \times n)$ matrix representing
constraint matrix and
- $B$ is a column vector of $(m+n-1)$ dimensional
represents the right hand side constraints.

Note that for any matrix $D$ of $r \times c$ dimensional,
the notation $D_{i,j}$, $i=1,2,\ldots,r$ will be used to denote the
$i$ th row, while, the notation $D_{c,j}$, $j=1,2,\ldots,c$ will be
used to denote the $j$ th column. This means that the
matrix $D$ can be written in row form as:
\[ D = [ D_{R1} \ D_{R2} \ \ldots \ D_{R_i} \ \ldots \ D_{R_r} ] \]

Also, the matrix \( D \) can be written in column form as:

\[ D = [ D_{C1} \ D_{C2} \ \ldots \ D_{C_j} \ \ldots \ D_{C_c} ] \]

Also, suppose that the index set of basic variables is denoted by \( I \) and the index set of nonbasic variables is denoted by \( J \) as follows:

\[ I = \{1, 2, \ldots, m+n-1\}, \quad J = \{1, 2, \ldots, m n - m - n + 1:\ i \neq j \text{ for all } i = 1, 2, \ldots, m+ n-1 \} \tag{9} \]

**Weighting Approach:** Several approaches for solving \( P(\text{MTP}) \) whether direct, scalarization, interactive or fuzzy approaches. The scalarization approach is the most common approach and the weighting approach is the most common one among them. Weighting approach can be used for transforming the multiobjective transportation problem \( P(\text{MTP}) \) into single transportation problem \( P(\lambda\text{TP}) \) as follows:

\[
P(\lambda\text{TP}): \quad \text{Min} \ [ \lambda \ \overline{C} \ \ X] \tag{10} \]

subject to:

\[
\begin{align*}
\Lambda \ X &= B \\
X, \ W \geq 0, \ W \ 1 &= 1
\end{align*} \tag{6} \tag{11}
\]

where,

\( \lambda \) is a row vector of \( K \)-dimensional representing the objective function weights,

\( \overline{C} \) is an \( K \times m \) matrix representing the objective function coefficients and

\( I \) is an identity column vector of \( K \) - dimensional.

For simplicity, let,

\[
C = \lambda \ \overline{C} \tag{12}
\]

Thus, the equivalent single transportation problem \( P(\lambda\text{TP}) \) is given by:

\[
P(\lambda\text{TP}): \quad \text{Min} \ [ \ C \ X] \tag{13}
\]

subject to:

\[
\begin{align*}
\Lambda \ X &= B \\
X, \ W \geq 0, \ W \ 1 &= 1
\end{align*} \tag{6} \tag{11}
\]

The optimum solution \( X^* \) of problem \( P(\lambda\text{TP}) \) is given by:

\[
X^* = [ x_{1*} \ x_{2*} \ \ldots \ x_{m*} ]^t \tag{13} \]

This optimum solution will be divided into two sets of variables, the set of \( m + n - 1 \) basic variables \( X^*_B \) that takes nonnegative values and the set of \( m - n + 1 \) non basic variables \( X^*_NB \) that takes the zero values as follows:

\[
X^*_B = [ x_{B1} \ x_{B2} \ \ldots \ x_{B m+n-1} ]^t \tag{14}
\]

\[
X^*_NB = [ x_{NB1} \ x_{NB2} \ \ldots \ x_{NB nm-n+1} ]^t = 0 \tag{15}
\]

Also, the initial objective function coefficients \( C^* \) will be divided into two sets of coefficients, the set of \( m+n-1 \) basic coefficients of the final optimal tableau \( C^*_B \) and the set of \( nm\ - mn + 1 \) non basic coefficients of the final optimal tableau \( C^*_NB \) and can be expressed as follows:

\[
C^*_B = [ C_{B1} \ C_{B2} \ \ldots \ C_{B m+n-1} ] \tag{16}
\]

\[
C^*_NB = [ C_{NB1} \ C_{NB2} \ \ldots \ C_{NB nm-n+1} ] \tag{17}
\]

Also, the initial matrix constraints coefficients \( A^* \) will be divided in column form into two sets, the set of \( m + n - 1 \) basic \( A^*_B \) that takes the identity value only at the corresponding basic variable while the others are zeros and the set of \( nm - m + n + 1 \) non basic \( A^*_NB \) that takes any values.

\[
A^*_B = [ A_{C1} \ A_{C2} \ \ldots \ A_{C B m+n-1} ] \tag{18}
\]

\[
A^*_NB = [ A_{C NB1} \ A_{C NB2} \ \ldots \ A_{C NB nm-n+1} ] \tag{19}
\]

While, in the final optimum simplex tableau the previous matrices will be defined as follows:

\[
\hat{B} = [ \hat{b}_1 \ \hat{b}_2 \ \ldots \ \hat{b}_{m+n-1} ] \tag{20}
\]

\[
\hat{C}_B = [ \hat{c}_{B1} \ \hat{c}_{B2} \ \ldots \ \hat{c}_{B m+n-1} ] \tag{21}
\]

\[
\hat{C}_{NB} = [ \hat{c}_{NB1} \ \hat{c}_{NB2} \ \ldots \ \hat{c}_{NB nm-n+1} ] \tag{22}
\]

\[
\hat{A}_B = [ \hat{A}_{C1} \ \hat{A}_{C2} \ \ldots \ \hat{A}_{CB m+n-1} ] \tag{23}
\]

\[
\hat{A}_{NB} = [ \hat{A}_{C NB1} \ \hat{A}_{C NB2} \ \ldots \ \hat{A}_{C NB nm-n+1} ] \tag{24}
\]

It is clear that an optimum solution of problem \( P(\lambda\text{TP}) \) is an efficient solution for problem \( P(\text{MTP}) \).

Also, note that if \( X^* \) is an efficient solution of problem \( P(\text{MTP}) \), then, there exists \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \) such that \( X^* \) is an optimal solution of problem \( P(\lambda\text{TP}) \).
Perturbed Problem: Perturbations to the right hand side constraints and to the objective function coefficients to the problem \( P(\lambda TP) \) can be viewed as a perturbed problem \( P(\lambda TP) \) as follows:

\[
P(\lambda TP): \text{Min} \left[ \sum_{j=1}^{n} \sum_{i=1}^{m} (1 + a_{ij}) c_{ij} x_{ij} \right]
\]

subject to:

\[
\sum_{j=1}^{n} \sum_{i=1}^{m} \beta_{ij} x_{ij} = \left(1 + \gamma_{i,j}\right) b_{j}, \quad j=1,2,\ldots,n
\]

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_{ij} = \left(1 + \lambda_{i,j}\right) a_{i}, \quad i=1,2,\ldots,m
\]

(Balanced condition)

\[
x_{ij} \geq 0, \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n
\]

where,

- \( a_{ij} \) are the multiplicative parameters of \( c_{ij} \) representing the allowable tolerance percentage of the objective function coefficients perturbations,
- \( \beta_{i} \) are the multiplicative parameters of \( a_{i} \) representing the allowable tolerance percentage of the supplies and
- \( \gamma_{j} \) are the multiplicative parameters of \( b_{j} \) representing the allowable tolerance percentage of the destinations.

The multiplicative parameters \( a_{ij} \) must not exceed a non negative number \( \tau \). While, the both multiplicative parameters \( \beta_{i} \) and \( \gamma_{j} \) must not exceed a non negative number \( \rho \).

\[
P(\lambda TP): \text{Min} \left[ (C \pm \Omega) X \right]
\]

subject to:

\[
\Lambda X = (B \pm \Delta)
\]

\[
X, \lambda > 0
\]

where,

\[
\Omega = [ a_{11}, a_{12}, \ldots, a_{mn} ]
\]

\[
\Lambda = [ \delta_{1}, \delta_{2}, \ldots, \delta_{m+n} ]^{t}
\]

Let,

- \( T^{*} \) is a multiplicative parameter of \( C \) that represents the maximum allowable tolerance percentage of the objective function coefficients perturbations and
- \( \delta^{*} \) is a multiplicative parameter of \( B \) that represents the maximum allowable tolerance percentage of the right hand side constraints perturbations.

Definition: (Critical Region of the Parameters): Suppose that the maximum allowable tolerance for the right hand side constraints perturbations \( \delta^{*} \) having the property that the same basis is optimal in the perturbed problem \( P(\lambda TP) \) as long as the absolute value of each parameter \( \delta_{i} \) satisfies the following:

\[
- \delta^{*} \leq \delta_{i} \leq \delta^{*}, \quad i \in I
\]

(25)

Also, suppose that the maximum allowable tolerance for the objective function coefficients perturbations \( T^{*} \) having the property that the same basis is optimal in the perturbed problem \( P(\lambda TP) \) as long as the absolute value of each parameter \( T_{ij} \) satisfies the following:

\[
- T^{*} \leq a_{ij} \leq T^{*}, \quad (i,j) \in J
\]

(34)

Thus, the critical region of the parameters of the objective functions coefficients and the right hand side constraints perturbations is denoted by \( R_{\lambda,a,\delta,c} \) and is defined as follows:

\[
R_{\lambda,a,\delta,c} = \{ \delta, \alpha, \lambda, C: - \delta^{*} \leq \delta_{i} \leq \delta^{*}, \quad i \in I \}
\]

(35)

The tolerance approach aims at finding this critical region of the parameters of both weights of the objectives, objective functions coefficients and the right hand side constraints at which any change inside their ranges of this region does not affect the optimal basis, while, any change outside their ranges will affect the optimal basis.

Theorem: (Wendell [9]): The value \( \delta^{*} \) that represents the maximum allowable tolerance for the right hand side constraints perturbations can be determined as follows:

\[
\delta^{*} = \min_{i \in I} \left\{ \frac{A_{i}^{*} B^{*}}{A_{i} B^{*}} \right\}, \quad i \in I
\]

(36)

While, the value \( \tau^{*} \) that represents the maximum allowable tolerance for the objective function coefficients perturbations can be determined as follows:

\[
\tau^{*} = \min_{(i,j) \in J} \left\{ \frac{C_{ij} A_{i}^{*} B^{*}}{A_{i}^{*} B^{*}} \right\}, \quad (i,j) \in J
\]

(37)

Note that all the information of \( \delta^{*} \) and \( T^{*} \) is readily available from the original and final optimal simplex tableau. Moreover, the critical region of the parameters of both weights of the objectives, objective functions coefficients and the right hand side constraints could be determined as follows:

\[
\begin{align*}
\delta^{*} &= \min_{i \in I} \left\{ \frac{A_{i}^{*} B^{*}}{A_{i} B^{*}} \right\}, \quad i \in I \\
T^{*} &= \min_{(i,j) \in J} \left\{ \frac{C_{ij} A_{i}^{*} B^{*}}{A_{i}^{*} B^{*}} \right\}, \quad (i,j) \in J
\end{align*}
\]
constraints at which any change inside their ranges of this region does not affect the optimal basis can be determined (35). The values of perturbed optimal basis \( \hat{Z}_p \) and perturbed objective value \( \hat{Z}_p \) can be determined as follows:

\[
\hat{B}_p = \hat{B} + \hat{A}_B \Delta
\]

\[
\hat{C}_p = \hat{C}_B - \Omega_{NB} + \hat{A}_MB \Omega_{NB}
\]

\[
\hat{Z}_p = \hat{Z} + \hat{C}_B \Delta + \hat{B}_p \Omega_B
\]

\[
\Delta = [ \delta_1, \delta_2, \ldots, \delta_{m+n-1} ]^T
\]

\[
\Omega_B = \{ \alpha_{i,j} | (i, j) \in I \}, \Omega_{NB} = \{ \alpha_{i,j} | (i, j) \in J \}
\]

where,

- \( \hat{B} \) is a column vector of \( m+n-1 \) dimensional represents the optimal basis variables in the final optimal simplex tableau,
- \( \hat{A}_B \) is a square matrix of \( m+n-1 \) dimensional represents the coefficients of initial basis in the final optimal simplex tableau,
- \( \Delta \) is a column vector of \( m+n-1 \) dimensional represents the allowable tolerance in the right hand side constraints perturbations as defined in the critical region,
- \( \Omega_B \) is a column vector of \( m+n-1 \) dimensional represents the allowable tolerance in objective function coefficients perturbations of basis variables,
- \( \Omega_{NB} \) is a column vector of \( mn-m+n-1 \) dimensional represents the allowable tolerance in the objective function coefficients perturbations of non basic variables and
- \( \hat{A}_MB \) is an \( m+n-1 \times mn-m+n-1 \) matrix represents the column coefficients constraints matrix of non basis variables in the final optimal simplex tableau.

5. Illustrative Example: Consider the multiobjective transportation problem P(MTP1) of two sources \( (m=2) \) and three destinations \( (n=3) \) as follows:

\[ P(MTP1) : \min [ C X ] \]

subject to:

\[ A X = B \]

\[ X \geq 0 \]

The data for the multiobjective transportation problem is given in Table 1, where the cell \((i, j)\) in the north-west corner gives the \( c_{i,j}^{(1)} \) while, the cell in the south-east corner gives the \( c_{i,j}^{(2)} \). Each row corresponds to a supply point and each column corresponds to a demand point. The total supply equals the total demand.

**Table 1:** Data for the multiobjective transportation problem.

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>6</td>
<td>3</td>
<td>0.5</td>
<td>90</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>90</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Demands: 60 60 60 Total = 180

**Table 2:** Initial simplex tableau for the illustrative example.

<table>
<thead>
<tr>
<th>( i, j )</th>
<th>( x_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_{1,1} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( u_{1,2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( u_{1,3} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( u_{1,4} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( (40) )</td>
<td>-3</td>
<td>-3</td>
<td>-1</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

**Table 3:** Final optimal simplex tableau for the illustrative example.

<table>
<thead>
<tr>
<th>( i, j )</th>
<th>( x_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( u_{i,j} )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{1,1} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_{2,1} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_{2,2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_{2,3} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( x_{3,1} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_{3,2} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_{3,3} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

\[ X = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix}^T \]

\[ A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 90 \\ 60 \\ 60 \end{bmatrix} \]

The multiobjective transportation problem P(MTP1) can be transformed into single transportation problem P(\( \lambda \)TP1) by using the weighting sum approach as follows:

\[ P(\lambda TP1) : \min [ C X ] \]

subject to:

\[ A X = B \]

\[ X, \lambda \geq 0, \quad \bar{X}^T \lambda = 1 \]

where,

\[ \lambda = [ \lambda_1, \lambda_2, \ldots, \lambda_3 ] \]

In this example, the weights of the objective functions are chosen by [0.6, 0.4] so that the resultant objective function coefficients \( C \) is given by:

\[ C = [ 4 \ 3 \ 2 \ 1 \ 3 \ 4 ] \]

The results of the original and final optimal simplex tableau for the example are given in Table 2 and Table 3.

5. Illustrative Example: Consider the multiobjective transportation problem P(MTP1) of two sources \( (m=2) \) and three destinations \( (n=3) \) as follows:

\[ P(MTP1) : \min [ C X ] \]

subject to:

\[ A X = B \]

\[ X \geq 0 \]

The data for the multiobjective transportation problem is given in Table 1, where the cell \((i, j)\) in the north-west corner gives the \( c_{i,j}^{(1)} \) while, the cell in the south-east corner gives the \( c_{i,j}^{(2)} \). Each row corresponds to a supply point and each column corresponds to a demand point. The total supply equals the total demand.
For simplicity, the decision variables $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, and $x_6$ will be denoted by numbers $1$, $2$, $3$, $4$, $5$, and $6$ respectively. While, the decision variables $u_1$, $u_2$, $u_3$, and $u_4$ will be denoted by numbers $7$, $8$, $9$, and $10$ respectively. Thus,

$$I = \{2, 3, 4, 5\}, \quad J = \{1, 6, 7, 8, 9, 10\}$$

$$A^* = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$A^* = \begin{bmatrix}
0 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix}$$

$$C^* = \begin{bmatrix}
3 & 2 & 1 & 3
\end{bmatrix}$$

$$C^* = \begin{bmatrix}
4 & 4 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The maximum allowable tolerance for the right hand side constraints perturbations $\delta^*$ of the problem $P(WP1)$ given by (37) and can be calculated as follows:

$$\delta^* = \max \left\{ \frac{1}{7}, \frac{1}{5}, \frac{1}{3} \right\} = \frac{1}{7} 14.29\%$$

Note that, for the balanced condition, the following perturbed condition must be satisfied:

$$\alpha_1 + \alpha_2 = \beta_1 + \beta_2 + \beta_3$$

(Balanced perturbed condition)

The maximum allowable tolerance for the objective function coefficients perturbations $T^*$ of the problem $P(\lambda TP1)$ given by (38) and can be calculated as follows:

$$\delta^* = \max \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{9}, \frac{1}{5} \right\} = \frac{1}{6} 11.11\%$$

The critical region of the parameters of the objective functions coefficients and the right hand side constraints perturbations $R_{\lambda, \alpha, \beta, \gamma}$ of the problem $P(\lambda TP1)$ given by (36) and is defined as follows:

$$R_{\lambda, \alpha, \beta, \gamma} = \{ \delta, \alpha, \lambda, C: \delta^* \leq \delta^*, \lambda^* \leq \lambda^*, \beta^* \leq \beta^*, \alpha^* \leq \alpha^*, \gamma \leq \gamma \}$$

The values of perturbed optimal basis $\tilde{B}_p$, perturbed objective values coefficients $\tilde{c}_p$, and perturbed objective value $\tilde{c}_{wp}$ of the problem $P(WP1)$ are given by (22-24) and can be determined as follows:

$$\tilde{B}_p = \begin{bmatrix}
31.43 & 81.43 & 51.43 & 81.43 & 51.43 & 81.43
\end{bmatrix}$$

$$\tilde{c}_p = \begin{bmatrix}
4.444 & 4.444 & 0 & 0 & 0
\end{bmatrix}$$

$$\tilde{c}_{wp} = \begin{bmatrix}
0.1429, 0.1111, 0, 0, 0, 0, 0, 0, 0, 0
\end{bmatrix}$$

The maximum allowable tolerance for the right hand side constraints perturbations $\delta^*$ of the problem $P(\lambda TP1)$ given by (37) and can be calculated as follows:

$$\delta^* = 14.29\% \quad T^* = 0.1111, \lambda^*, \beta^*, \gamma^* = \frac{11}{9}$$
\[ \hat{Z}_{\rho} = 360 + [2 2 -1 1] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} + (60+\delta_2-\delta_3-\delta_4) \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} \]

where,
\[ \Delta = [\delta_1, \delta_2, \delta_3, \delta_4] \]
\[ \Omega_{B} = [a_2, a_3, a_4, a_5] \]
\[ \Omega_{NB} = [a_1, a_6, a_7, a_8, a_9] \]
\[ C^*_{B} = [3 2 1 3], C^*_{NB} = [4 4 0 0 0] \]

\[ \hat{A}_{B} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \hat{A}_{NB} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \]

\[ (59) \]

**Conclusion:** This paper presents the tolerance approach to sensitivity analysis in multiobjective transportation problem. The weighted sum approach is used to convert the multiobjective transportation problem into single transportation problem. The proposed approach allows changing in both the weights, objective functions coefficients and in the right hand side terms simultaneously and independently from their specified values while remaining the same solution optimal. The proposed approach can be considered as an extension of Wendell’s tolerance approach in a special case of linear programming and in the case of multiobjective transportation problem. Also, the proposed approach can be considered as another tolerance approach of both Hansen, Labbe and Wendell’s tolerance approach in a special case of the transportation problem that allows changing in both the weights, the objective functions coefficients and in the right hand side terms simultaneously and independently from their specified values while remaining the same solution optimal. The condition of balanced transportation problem is taken into consideration when dealing with the right hand side constraints. The proposed approach differs from their approach in allowing both weighting and objective functions coefficients to change parametrically according to maximum certain specified tolerance intervals while remaining the same solution optimal. The proposed tolerance approach aims at finding the balanced region of the parameters of both weights of the objectives, objective functions coefficients and the right hand side constraints at which any change inside their ranges of this region does not affect the optimal basis. The proposed tolerance approach uses only the data of initial and final simplex tableau of the equivalent single linear problem. Formulation of both obtained perturbed solution and the corresponding perturbed objective values are presented. An illustrative example is presented to clarify the idea of the proposed approach.

**REFERENCES**