Journal of Applied Sciences Research, 2(10): 751-752, 2006

Some Remarks on Normal Structure

J.O. Olaleru

Department of Mathematics, University of Lagos, Lagos, Nigeria.

Abstract: The concept of normal structure (introduced by Brodskii and Milman) is generally and commonly used as a condition for determining the existence of fixed points (or common fixed points) of nonexpansive maps (or commutative family of nonexpansive maps) defined on a closed, convex and bounded subset K of a reflexive Banach space X. In this paper, we show that the direct sum of metrisable topological vector space, each with normal structure, itself has normal structure. This is a generalization of the result of Belluce *et al*.

Key words: Normal structure, metrisable topological vector spaces.

INTRODUCTION

Let C be a bounded subset of a metric space X. The diameter of C is defined by

$$D(C) = \sup\{d(x, y): x, y \in C\}$$

A point $x_o \in X$ is called a non diametral point of C if

$$\sup\{d(x_o, y): x, y \in C\} < D(C)$$

Let D(K) denote the diameter of K. Then a bounded, convex subset K of X is said to have a *normal structure* if every non-trivial convex subset C of K contains at least one non diametral point. The metric space X itself is said to have a normal structure if every bounded, convex subset of X has normal structure. Geometrically, K is said to have a normal structure if for every non-trivial convex subset C of K there exists a ball centred at a point of C and whose radius is less than the diameter of C such that the ball contains C, i.e., if C is an arbitrary non-trivial convex subset of K, then $C \subseteq B(x_o, r)$ for some $x_o \in D(C)$. See Chidume[4].

A uniformly convex Banach space and a compact convex subset of a Banach space each has a normal structure, Brosdskii[3], and every compact convex subset of a metrisable locally convex space has a normal structure too, Olaleru[8].

The concept of normal structure was introduced by Brodskii *et al*.[3] and was used by Kirk[6] to prove the existence of a fixed point of a nonexpansive map defined on a closed convex and bounded subset of a reflexive Banach space. The author recently generalized this result to a reflexive metrisable locally convex space, Olaleru[9]. It was proved by Belluce *et al*.[1] that if K is a nonempty, closed, convex and weakly subset of a Banach space such that K has a normal structure; if Γ is a finite family of commutative nonexpansive mappings T on K, then there is a point $x \in K$ such that $T(x) = x$ for each $f \in \Gamma$. Belluce *et al.*[2] proved the following theorem.

Theorem A:[2] Let B_1 and B_2 be Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Let $B = B_1 \oplus B_2$ with the norms of B given by $\|\cdot\| = \sup(\|\cdot\|_1, \|\cdot\|_2)$. If Both B_1 and B_2 have normal structure, then B has normal structure.

We want to show, using the same technique of Belluce at al, that this theorem can be generalized to metrisable topological vector spaces. Metrisable topological vector spaces include metrisable locally convex spaces and normed spaces, e.g. see Iyahen[5], Robertson *et al.*[11] and Schaffer[12]. For recent results on fixed point theory in both metrisable locally convex spaces and metrisable topological vector spaces see Olaleru[10].

The main theorem: The following result is fundamental to the proof of our theorem.

Lemma[7,15.11]. The topology of a metrisable topological vector space X can always be defined by a real-valued function on X called *F-norm* satisfying

1. $F(x) > 0$,
2. $F(x) = 0 \to x = 0$,
3. $F(x+y) \leq F(x) + F(y)$,
4. $F(\alpha x) \leq F(x)$ for all α in *R* with $\alpha \leq 1$ and
5. If $\alpha_n \to 0$ and $\alpha_n \in K$, then $\alpha_n x \to 0$ for all $x, y \in X$.

Observe that if X is a Banach space, then the *F-norm* reduces to a norm.

Theorem: Let $X_1, X_2 \ldots.. X_n$ be metrisable topological vector spaces with metrics defined by F-norms $F_1, F_2 \ldots.. F_n$, respectively. Let $X = X_1 \oplus X_2 \oplus \ldots\ldots X_n$ such that

Corresponding Author: J.O. Olaleru, Department of Mathematics, University of Lagos, Lagos, Nigeria.
E-mail: olalerul@yahoo.com

the metric of X is defined by F-norm $F = \max (F_1, F_2, \ldots F_n)$. If each $X_1, X_2 \ldots X_n$ have normal structure, then X has normal structure.

Proof: It is sufficient to proof the case for $X = X_1 \oplus X_2$. Let K be a bounded convex subset of X which contains more than one point and let p_i be the natural projection of X onto X_i, i = 1, 2. Let $K_i = p_i(K)$, i = 1, 2. Then K_i is a bounded, convex subset of X_i and thus possess normal structure by assumption and definition. For i = 1, 2, let x_i be a nondiametral point of K_i. (If K_i consists of a single point for some i, then the conclusion of the theorem is immediate).

Select $u_i \in p_i^{-1}(x_i) \cap K$, i= 1, 2; then $u_1 = x_1 \oplus v$ and $u_2 = w \oplus x_2$ where $v = p_2(u_1) \in K_2$, $w = p_1(u_2) \in K_1$. Set $m = ½(u_1 + u_2) = m_1 \oplus m_2$ where $m_1 = ½(x_1 + w)$ and $m_2 = ½(v + x_2)$. Let $z \in K$; then $z = z_1 \oplus z_2$ where $z_1 \in K_1$ and $z_2 \in K_2$. Now $F(m - z) = \max (F_1(m_1 - z_1), F_2(m_2 - z_2))$. But $F_1(m_1 - z_1) = F_1(½(x_1 + w) - ½(z_1 + z_1)) \leq ½(F_1(x_1 - z_1) + F_1(w - z_1))$. Denote the diameter of a set K by D(K), and using the fact that x_1 is a nondiamtral point of K_1 we see that $F_1(m_1 - z_1) \leq D(K_1) - \varepsilon_1$ for some $\varepsilon_1 > 0$ which does not depend on z_1. Similarly we see that $F_2(m_2 - z_2) \leq D(K_2) - \varepsilon_2$, $\varepsilon_2 > 0$ and independent of z_2. Hence $F(m - z) \leq \max (D(K_1) - \varepsilon_1, D(K_2) - \varepsilon_2)$; Letting $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ we see that $F(m - z) \leq \max (D(K_1) - \varepsilon, D(K_2) - \varepsilon) = D(K) - \varepsilon$. Thus m is a nondiametral point for K and the theorem is proved.

Corollary 1: Let $X_1, X_2 \ldots X_n$ be metrisable locally convex spaces with metrics defined by F-norms $F_1, F_2 \ldots F_n$ respectively. Let $X = X_1 \oplus X_2 \oplus \ldots X_n$ such that the metric of X defined by F-norm $F = \max (F_1, F_2, \ldots F_n)$. If each $X_1, X_2 \ldots X_n$ have normal structure, then X has normal structure.

Corollary 2: Let $X_1, X_2 \ldots X_n$ be Banach spaces with metrics defined by norms $\|\cdot\|_1, \|\cdot\|_2 \ldots \|\cdot\|_n$ respectively. Let $X = X_1 \oplus X_2 \oplus \ldots X_n$ with the norm of X defined by $\|\cdot\| = \max (\|\cdot\|_1, \|\cdot\|_2, \ldots \|\cdot\|_n)$. If each $X_1, X_2 \ldots X_n$ have normal structure, then X has normal structure.

Remark: Generalising the result of our main theorem to infinite number of metrisable topological spaces still remains an open problem.

REFERENCES

1. Belluce, L.P. and W.A. Kirk, 1966. Fixed point theorems for families of contraction mappings. Pacific J. Math., 18: 213-218.
2. Belluce, L.P., W.A. Kirk and E.F. Steiner, 1968. Normal structure in Banach spaces, Pacific J. Math., 26(3): 433-440.
3. Brosdskii, M.S. and D.P. Mil'man, 1948. On the centre of a convex set, Dokl. Akad. Nauk. SSSR, 5: 837-840.
4. Chidume, C.E., 1996. Applicable Functional Analysis, International Centre for Theoretical Physics, Italy, 1996.
5. Iyahen, S.O., 1998. The Closed graph theorem, Osaruwa Educational Publications, Nigeria.
6. Kirk, W.A., 1965. A fixed point theorem for mappings which do not increase distance, Amer. Math. Monthly 72: 1004-1006.
7. Kothe, G., 1969. Topological Vector Spaces I, Springer-Verlag, 1969.
8. Olaleru, J.O., 2006. On the convergence of the Mann iteration in locally convex spaces, to appear in Carpathian J. Math, Vol 21, 2006.
9. Olaleru, J.O., 2006. Fixed points of quasi-contraction maps in topological vector spaces, submitted
10. Olaleru, J.O., 2006. On Kannan maps in locally convex spaces, Proceedings of the International Conference on "New trends in mathematical and Computer sciences with applications to real world problems", 14-19th June 2006, Covenant University, Cannanland, Nigeria, pp: 573-585.
11. Robertson, A.P., W.J. Robertson, 1980. Topological vector spaces, Cambridge University Press.
12. Schaffer, H.H., 1999. Topological vector spaces, Springer-Verlag.