Index Tracking Modeling in Portfolio Optimization with Mixed Integer Linear Programming

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ABSTRACT

Background: Index tracking is a portfolio management which aims to construct an optimal portfolio to generate similar return with the market index return with only selecting few numbers of stocks from the market index. In index tracking, the investors do not have to purchase all the stocks exactly in the market index because the transaction cost incurred is very high. Therefore, index tracking has been introduced to achieve similar return with the market index return without purchasing all the stocks that make up the index using mathematical model. Objective: The objective of this paper is to determine the performance of the optimal portfolio constructed with different number of stocks selected from the benchmark market index using mixed integer linear programming model in index tracking. The portfolio performance is determined in terms of tracking error. In this study, the data consists of 24 component stocks in Malaysia market index which is FTSE Bursa Malaysia Kuala Lumpur Composite Index from January 2010 until December 2012. Results: The result shows that the tracking error for the optimal portfolios constructed with different percentage of stocks selected from the index are closer to zero. This implies that the optimal portfolio constructed using mixed integer linear programming model is able to track the market index effectively in Malaysia. Conclusion: The optimal portfolio constructed using mixed integer linear programming model is able to generate similar return with the market index return effectively without purchasing all the stocks that make up the index. Therefore, the mixed integer linear programming model is suitable for the investors in Malaysia.

INTRODUCTION

In investment, the investors and fund managers aim to generate return at low risk by holding few numbers of stocks in their portfolios. Risk is the chance that an investment’s return will be different from the expected return. Markowitz (1952) introduced mean-variance model to determine the optimal portfolio at minimum risk. In this model, variance has been used as risk measure in portfolio optimization. Roll (1992) introduced index tracking in portfolio management using tracking error variance model. Index tracking is a type of investment which aims to construct an optimal portfolio to generate similar return with the stock market index return without purchasing all the stocks in the market index. In fact, the investors can purchase all stocks in the market index exactly with same proportions. However, this strategy may incur high transaction cost. Therefore, index tracking has been introduced for the investors to achieve similar return with the market index return with only holding some of the stocks in the market index. Roll (1992) applied variance as risk measure for tracking error. Tracking error is a chance that the actual return of the tracking portfolio is different from the return of the market index. Alexander (2005) proposed cointegration model in constructing the optimal portfolio in index tracking. The cointegration model has been studied by different researches in different stock market index. (Subramaniam, 2008, and Grobys, 2011). Canakgoz and Beasley (2009) presented a mixed integer linear programming model which adopts regression approach in index tracking. The objective of this paper is to determine the performance of the optimal portfolio constructed with holding different number of stocks using mixed integer linear programming model in index tracking. The portfolio performance is determined in terms of tracking error. The rest of the paper is organized as follows. The next section describes the data and methodology. Section 3 discusses about the empirical results of this study. Section 4 concludes the paper.
Data and Methodology:
In this study, the data consists of weekly price of 24 stocks in FTSE Bursa Malaysia Kuala Lumpur Composite Index (FBMKLCI) from January 2010 until December 2012. These 24 stocks are selected in the study since they make up as components of FBMKLCI consistently within the study period. The optimal portfolio is constructed with different percentage of stocks selected from the index using mixed integer linear programming model. The percentages of stocks selected from the index are 30%, 40%, 50%, 60% and 70% in this study (Jansen and Dijk, 2002). The optimal portfolio performance is measured in terms of tracking error. The optimal portfolio is computed using LINGO and EVIEWS software.

Mixed Integer Linear Programming Model:
Canakgoz and Beasley (2009) introduced mixed-integer programming model that adopts ordinary least square regression approach in index tracking. Regression is a statistical methods used to describe the nature of the relationship between the variables. The regression model is given as below when a linear regression is performed on the return from the tracking portfolio against the return from the market index,

\[ r_t = \alpha + \beta R_t \]  

The ordinary least-square estimates, \( \hat{\alpha} \) and \( \hat{\beta} \), for the intercept and slope respectively of the regression line are given as below.

\[ \hat{\alpha} = \sum_{i=1}^{N} w_i \hat{\alpha_i} \]  
\[ \hat{\beta} = \sum_{i=1}^{N} w_i \hat{\beta_i} \]

where \( w_i \) is the weight of stock \( i \) in the tracking portfolio, \( \hat{\alpha_i} \) and \( \hat{\beta_i} \) are the ordinary least-squares regression intercept and slope respectively when regression is performed on the returns from stock \( i \) against the index returns \( R_t \). In order to track the market index perfectly, the target values for both estimates are \( \hat{\alpha} = 0 \) and \( \hat{\beta} = 1 \). The model aims to achieve the desired intercept of zero and desired slope of one. Therefore, the objective of the model is to minimize \( |\alpha - 0| + |\beta - 1| \). The model is formulated as below.

Minimize \( \lambda_\alpha |\alpha - 0| + \lambda_\beta |\beta - 1| \)

Subjects to

\[ \sum_{i=1}^{K} z_i = K \quad (4) \]
\[ \sum V_i x_i = C \quad (5) \]
\[ w_i = \frac{V_i x_i}{C} \quad (6) \]
\[ \sum_{i} w_i = 1 \quad (7) \]
\[ L_i z_i \leq w_i \leq U_i z_i \quad (8) \]
\[ z_i \in [0,1] \quad (9) \]
\[ x_i, w_i \geq 0 \quad (10) \]

where \( K \) is number of stocks selected to track the stock market index, \( L_i \) and \( U_i \) are the lower and upper bounds of the investment proportion respectively on stock \( i \), \( V_{iT} \) is the price of one unit of stock \( i \) at time \( T \), \( x_i \) is the number of units of stock \( i \) in tracking portfolio and \( C \) is the total amount invested at time \( T \). The objective function is in modulus form which is nonlinear. The modulus objective function can be linearized by introducing variable \( D \) and \( E \). The mixed integer linear programming model is formulated as below.

Minimize \( \lambda_\alpha D + \lambda_\beta E \)

Subjects to
\[ \sum_{i=1}^{N} z_i = K \]  
(11)

\[ \sum V_{it}x_i = C \]  
(12)

\[ w_i = \frac{V_{it}x_i}{C} \]  
(13)

\[ \sum_{i=1}^{K} w_i = 1 \]  
(14)

\[ L_i z_i \leq w_i \leq U_i z_i \]  
(15)

\[ z_i \in [0,1] \]  
(16)

\[ D \geq \alpha \hat{E} \]  
(17)

\[ D \geq -\alpha \hat{E} \]  
(18)

\[ E \geq \beta - 1 \]  
(19)

\[ E \geq -\left(\beta - 1\right) \]  
(20)

\[ x_i, w_i, D, E \geq 0 \]  
(21)

The linear objective function is weighted between the desired slope of one and desired intercept of zero where \( \lambda_{\alpha} + \lambda_{\beta} = 1 \). In this study, both values are set as equally important, \( \lambda_{\alpha} = \lambda_{\beta} = 0.5 \). Constraint (11) ensures that the number of stocks in the tracking portfolio equals to \( K \). In this study, \( K \) is set to 9, 12, 15, 18 and 21 which are 30%, 40%, 50%, 60% and 70% respectively of the components in the market index. Constraint (12) defines the total amount invested at time \( T \) for the tracking portfolio. Constraint (13) defines the weights of stock \( i \) in tracking portfolio. Constraint (14) ensures that the total weights of stocks invested are one. For constraints (15) and (16), \( z_i (i = 1,2,\ldots,N) \) is introduced to indicate the stock selection problem with \( z_i = 1 \) indicates the \( i \)th stock is included in the tracking portfolio or otherwise for \( z_i = 0 \). Constraints (17) – (20) linearize the modulus form of the objective function. Constraint (21) ensures \( x_i, w_i, D, E \) are non-negative.

**Tracking Error:**

Index tracking aims to construct an optimal portfolio to achieve similar return with the market index return without purchasing all the stocks that make up the index. Variance is the most commonly used risk measure in portfolio optimization (Markowitz, 1952). Tracking error is the standard deviation of the difference between the returns of the tracking portfolio and the returns of the stock market index (Meade and Salkin, 1990). Tracking error has been used as a risk measure in different index tracking models (Roll, 1992, Beasley et al., 2003, Wu et al., 2007, Lam et al., 2013). The formula for the tracking error is as follows.

\[ TE = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (R_{pt} - R_{nt})^2} \]  
(22)

where \( TE \) is the tracking error, \( n \) is the number of periods, \( R_{pt} \) is the mean return of the portfolio at time \( t \) and \( R_{nt} \) is the mean return of the stock market index at time \( t \). A portfolio is considered tracking the stock market index perfectly if there is zero tracking error. This implies that the tracking portfolio is closer to the market index at lower tracking error. The optimal level of tracking error is below 3% (Fabozzi et al., 2002). The performance of the optimal portfolio constructed by mixed integer linear programming model is compared to the market index in terms of tracking error.

**Results:**

Table 1 presents the performance of the optimal portfolio constructed at different number of stocks selected from FBMKLCI index. As shown in Table 1, the optimal portfolios are constructed with different percentage of stocks selected from FBMKLCI index which consist of 30 stocks using mixed integer linear programming model. The tracking errors are same (0.50%) for the optimal portfolios with 30%, 40% and 50% of stocks selected from FBMKLCI index. This implies that the investors can select smaller number of stocks which is 30% of FBMKLCI index components in their portfolios since the transaction cost is lower. The tracking error for the optimal portfolios...
with 60% and 70% of stocks selected from the index are 0.46% and 0.42%. The tracking error decreases when the percentage of stocks selected from the market index increases to more than 50% since the risk is getting lower. The result shows that the tracking error for the optimal portfolios constructed at different percentage of stocks selected from the index is closer to zero and well below 3.00%. This implies that the optimal portfolio constructed using mixed integer linear programming model is able to track FBMKLCI index effectively.

**Table 1**: Optimal portfolio performance at different number of stocks selected from FBMKLCI index.

<table>
<thead>
<tr>
<th>Percentage of stocks selected from benchmark FBMKLCI index (%)</th>
<th>Number of stocks selected in portfolio (K)</th>
<th>Tracking Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>9</td>
<td>0.50</td>
</tr>
<tr>
<td>40</td>
<td>12</td>
<td>0.50</td>
</tr>
<tr>
<td>50</td>
<td>15</td>
<td>0.50</td>
</tr>
<tr>
<td>60</td>
<td>18</td>
<td>0.46</td>
</tr>
<tr>
<td>70</td>
<td>21</td>
<td>0.42</td>
</tr>
</tbody>
</table>

**Conclusion:**

This paper discusses about the performance of the optimal portfolio constructed using mixed-integer linear programming model to track FBMKLCI index with different number of stocks from the index. The tracking error for the optimal portfolio constructed with different number of stocks is closer to zero. In conclusion, the mixed integer linear programming model is able to track FBMKLCI index effectively. Therefore, the model is suitable for the investors in Malaysia.

**REFERENCES**


