A New Model for Obtaining the Binding Energy of Lithium Isotopes Nucleus in the Woods-Saxon Potential

1M. Ghazvini, 2N. Salehi, 1A.A. Rajabi

1Department of Physics, Shahrood University, Shahrood, Iran.
2Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran.

ABSTRACT

In this work, we calculated the ground state binding energy of Lithium nucleus. For this purpose, first we studied the Schrödinger equation for 6-body system and we solved this equation by using the Pekeris approximation and supersymmetry method for the Woods-Saxon potential. Then by applying our proposed model for Lithium isotopes nucleus we calculated the binding energy of them. The results of our model for all calculations show that the ground state binding energy of Lithium isotopes nucleus with this potential is very close to the ones obtained in experiments.

INTRODUCTION

In quantum mechanics, the Schrödinger equation is a partial differential equation that describes how the quantum state of some physical system changes with time. It was formulated in late 1925, and published in 1926, by the Austrian physicist Erwin Schrödinger. The quantum N-body problem has been posed since 1926 in a precise mathematical form: the Schrödinger equation for N particles interacting pairwise by two-body potentials which vanish at infinity. Together with the general principles of quantum mechanics this equation represents the simple, unifying basis for understanding all forms of nonrelativistic matter from the atomic point of view. Of course spin and statistics as well as the coupling to electromagnetic fields must be included to substantiate this claim, but these aspects will not be considered in our review.

Until now, some work is done to solve Schrödinger equation for example for 2, 3-body particles (Salehi, N., A.A. Rajabi, 2012; Salehi, N., A.A. Rajabi, 2009). Now we want to do it for 6 to 11-body particles, for instance lithium element and their isotopes. There is several ways to do it. NU method and Supersymmetry (SUSY) method are two of them that we introduce supersymmetry method (Salehi, N., A.A. Rajabi, 2012). In this work, we have presented the exact solution of Schrödinger equation with the Woods-Saxon potential by using the SUSY method with Pekeris approximation (Panella, O., S. Biondini, 2010; Bayrak, O., G. Kocak, 2006).

1. Woods – Saxon Potential:

The Woods-Saxon potential is a reasonable potential for nuclear shell model and hence attracts lots of attention in nuclear physics. The Woods-Saxon potential plays an essential role in microscopic physics, since it can be used to describe the interaction of a nucleon with the heavy nucleus (Pahlavani, M.R., J. Sadeghi, 2009; Woods, R.D., D.S. Saxon, 1954). Although the non-relativistic Schrödinger equation with this potential has been solved for ground state and the single particle motion in atomic nuclei has been explain quite well, the relativistic effects for a particle under the action of this potential are more important, especially for a strong-coupling system. The Woods – Saxon potential as follows:

\[ V = \frac{-V_o}{1 + \exp \left( \frac{r - R_s}{a} \right)} \]

where \( V_o \) is the potential depth, \( R_s \) is the width of the potential and \( a \) is its diffuseness and \( a \) is
the surface thickness which is usually adjusted to the experimental values of ionization energies.

2. **Supersymmetry Method:**

We start noticing that we know the ground state function of a 1D problem; we can find also the potential, up to a constant. Taking the ground state energy to be zero, from the TISE we have

\[ H_1 \Psi_0(x) = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_0(x)}{dx^2} + V(x) \Psi_0(x) = 0 \]  

where \( V_1 \) as follows:

\[ V_1 = \frac{\hbar^2}{2m} \frac{d^2 \Psi_0(x)}{dx^2} \]  

We can try to factorize the Hamiltonian with the ansatz

\[ H_1 = A^2 A = H - E_0 \]  

where

\[ A = \frac{\hbar}{\sqrt{2m}} dW(x), A^* = -\frac{\hbar}{\sqrt{2m}} dW(x) \]  

and \( W(x) \) is called the super potential and can be written in terms of the ground state as:

\[ W(x) = -\frac{\hbar}{\sqrt{2m}} \Psi'_0 \]  

By writing \( V_1 \) in terms of \( W(x) \) we obtain the Ricatti equation:

\[ V_1(x) = W'(x) - \frac{\hbar}{\sqrt{2m}} W(x) \]  

So now we can build up a SUSY theory searching for the SUSY partner Hamiltonian associated to \( H_1 \), namely \( H_2 = AA^* \). This second Hamiltonian corresponds to a new potential:

\[ H_2 = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \Psi_0(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W(x) \]  

3. **Exact Analytical Solution of the Schrödinger Equation for Woods – Saxon Potential:**

The Schrödinger equation in D-dimension is

\[ \left( \frac{d^2 \Psi}{dr^2} + \frac{D-1}{r} \frac{d \Psi}{dr} - \frac{\gamma (r + D - 2)}{r} \right) R(r) + V(r) R = ER \] \hspace{1cm} \[ D = 3N - 3 \]  

\( N \) is the number of particle and \( \gamma \) is the grand angular momenta (Hasanabadi, H., A.A. Rajabi, 2009). First, we want to solve the Schrödinger equation for 6-body system that gives us 15- dimension. So we reach to:

\[ \frac{d^2 U}{dr^2} + \frac{2\mu}{\hbar^2} \left( E - V(r) \right) \frac{h^2}{2\mu r^2} \left( l + 6 \right) \left( l + 7 \right) U = 0 \]  

where \( l \) is the angular momentum quantum number and \( \mu \) is the reduced mass. It is suitable to introduce the woods-Saxon potential in the form of

\[ V = \frac{-V_0}{1 + \exp \left( \frac{r - R_0}{a} \right)} \]  

By putting this potential (Eq.11) into Eq. (10), it takes the form:

\[ \frac{d^2 U}{dr^2} + \frac{2\mu}{\hbar^2} \left( E - V \right) \frac{h^2}{2\mu r^2} \left( l + 6 \right) \left( l + 7 \right) U = 0 \]  

We define the effective potential in Eq. (12) as the following form:

\[ V_{\text{eff}} = \frac{V_0}{1 + \exp \left( \frac{r - R_0}{a} \right)} - \frac{h^2 (l + 6) (l + 7)}{2\mu a^2} \]  

By applying \( r = R_0(x + 1), \frac{R_0}{a} = a \) and the expansion \((x + 1)^2 = 1 - 2x + 3x^2 + ...\) we found the following form for Eq. (13):

\[ V_{\text{eff}} = \frac{V_0}{1 + \exp(a x)} - \delta(1 - 2x + 3x^2 + ...) \]  

where \( \delta \) is defined as:
\[ \delta = \frac{\hbar^2 (l+6)(l+7)}{2 \mu R_0^2} \]  

(15)

According to the Pekeris approximation, we can substitute effective potential \( V_{\text{eff}} \) with Eq. (16):

\[ V_{\text{eff}} = \frac{V_0}{1+\exp(\alpha x)} + \delta \left( d_0 + \frac{d_1}{2} + \frac{d_2}{4} \right) \]  

(16)

We use the Taylor expansion for the exponential terms of the right side of Eq. (16) and compare it with Eq. (14) we get:

\[ \left( d_0 + \frac{d_1}{2} + \frac{d_2}{4} \right) = 1 \quad \frac{\alpha}{4} (d_1 + d_2) = 2 \quad \frac{\alpha^2}{16} d_3 = 3 \]  

(17)

It is three equations with three variables. By solving these equations, \( d_0, d_1 \) and \( d_2 \) will be:

\[ d_0 = 1 + \frac{12}{\alpha^2} \quad d_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2} \quad d_2 = \frac{48}{\alpha^2} \]  

(18)

By using Eq. (16) into Eq. (12), the Schrödinger equation changes to:

\[ \frac{d^2 U}{dx^2} + \left( \frac{\beta}{1+\exp(\alpha x)} \right) U = 0 \]  

(19)

By choosing:

\[ \varepsilon = \frac{2\mu}{\hbar} (E - \delta d_0) \quad \beta = \frac{2\mu}{\hbar} (V_0 - \delta d_1) \quad \gamma = \frac{2\mu}{\hbar} \delta d_2 \]  

(20)

we will have:

\[ \frac{d^2 U}{dx^2} + \left( \varepsilon + \frac{\beta}{1+\exp(\alpha x)} - \frac{\gamma}{(1+\exp(\alpha x))^2} \right) U = 0 \]  

(21)

In Supersymmetric Quantum Mechanics, the superpotential is defined as:

\[ W_\alpha = \frac{\hbar}{\sqrt{2\mu}} \left[ A + \frac{B}{1+\exp(\alpha x)} \right] \]  

(22)

By substituting this superpotential into the Riccati Equation that has the form of:

\[ W^2_\alpha(x) - W'(x) = \frac{2\mu}{\hbar} (V_\alpha(x) - E_{\alpha}(1)) \]  

(23)

we will have:

\[ \left( A^2 + \frac{B^2 - \alpha B}{1+\exp(\alpha x)} \right) - \frac{2AB + B\alpha}{1+\exp(\alpha x)} \right) = -\varepsilon + \frac{\beta}{1+\exp(\alpha x)} + \frac{\gamma}{(1+\exp(\alpha x))^2} \]  

(24)

So by doing some calculations, we can get \( A^2 = -\varepsilon \cdot 2AB - \alpha B = -\beta \cdot B^2 + B\alpha = \gamma \) and the ground state binding energy for Lithium nucleus is given as following:

\[ E_{\alpha} = \frac{\hbar^2}{2\mu} \left( \frac{\beta}{\alpha \pm \sqrt{\alpha^2 + 4\gamma}} + \frac{\alpha}{2} \right) + \delta d_0 \]  

(25)

By using Eq. (26) from SUSY method:

\[ \psi_\alpha(x) = N_\alpha \exp \left( -\int W(x) dx \right) \]  

(26)

the ground state normalized eigenfunctions are given as:

\[ \psi_\alpha(x) = N_\alpha \exp \left( \frac{\hbar}{\sqrt{2\mu}} \left( A x + \frac{B}{\alpha} \ln \frac{\exp(\alpha x)}{1+\exp(\alpha x)} \right) \right) \]  

(27)

In a similar manner, we can applying our proposed model for other Lithium isotopes. So the ground state binding energy for \(^7\text{Li}\) is obtained as:

\[ E_{\alpha} = \frac{\hbar^2}{2\mu} \left( \frac{\beta}{\alpha \pm \sqrt{\alpha^2 + 4\gamma}} + \frac{\alpha}{2} \right)^2 + \delta d_0 \quad, \quad \delta = \frac{\hbar^2 (2l+15)(2l+17)}{8\mu R_0^2} \]  

(28)

and the ground state binding energies for \(^6\text{Li}, ^7\text{Li}, ^8\text{Li}\) and \(^11\text{Li}\) are respectively:

\[ E_{\alpha} = \frac{\hbar^2}{2\mu} \left( \frac{\beta}{\alpha \pm \sqrt{\alpha^2 + 4\gamma}} + \frac{\alpha}{2} \right)^2 + \delta d_0 \quad, \quad \delta = \frac{\hbar^2 (l+9)(l+10)}{2\mu R_0^2} \]  

(29)
\[ E_{n\alpha} = -\frac{\hbar^2}{2\mu} \left( \frac{-\beta}{\alpha \pm \sqrt{\alpha^2 + 4\alpha'}} + \frac{\alpha}{2} \right) + \delta_d \quad , \quad \delta_d = \frac{\hbar^2 (2l + 21)(2l + 23)}{8\mu R_e^2} \]  

\[ E_{n\alpha} = -\frac{\hbar^2}{2\mu} \left( \frac{-\beta}{\alpha \pm \sqrt{\alpha^2 + 4\alpha'}} + \frac{\alpha}{2} \right) + \delta_d \quad , \quad \delta_d = \frac{\hbar^2 (d + 12(l + 13))}{2\mu R_e^2} \]  

\[ E_{n\alpha} = -\frac{\hbar^2}{2\mu} \left( \frac{-\beta}{\alpha \pm \sqrt{\alpha^2 + 4\alpha'}} + \frac{\alpha}{2} \right) + \delta_d \quad , \quad \delta_d = \frac{\hbar^2 (2l + 27)(2l + 29)}{8\mu R_e^2} \]  

In Table 1 the fitted values of parameters of the ground state binding energy equations are given.

<table>
<thead>
<tr>
<th>Lithium Isotopes</th>
<th>( a ) (fm)</th>
<th>( R ) (fm)</th>
<th>( V_d ) (MeV)</th>
<th>B.E (MeV) Our model</th>
<th>B.E (MeV) Expriment [16]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^6\text{Li})</td>
<td>0.48</td>
<td>2.499</td>
<td>32.71</td>
<td>31.990560</td>
<td>31.995</td>
</tr>
<tr>
<td>(^7\text{Li})</td>
<td>0.48</td>
<td>2.661</td>
<td>50</td>
<td>39.339590</td>
<td>39.25</td>
</tr>
<tr>
<td>(^9\text{Li})</td>
<td>0.48</td>
<td>2.796</td>
<td>49.15</td>
<td>41.279710</td>
<td>41.278</td>
</tr>
<tr>
<td>(^9\text{Li})</td>
<td>0.48</td>
<td>2.918</td>
<td>49.08</td>
<td>45.383970</td>
<td>45.342</td>
</tr>
<tr>
<td>(^10\text{Li})</td>
<td>0.46</td>
<td>2.942</td>
<td>50</td>
<td>45.649430</td>
<td>45.318</td>
</tr>
<tr>
<td>(^11\text{Li})</td>
<td>0.43</td>
<td>2.9</td>
<td>32.5</td>
<td>45.625830</td>
<td>45.642</td>
</tr>
</tbody>
</table>

**Conclusion:**

In this paper, we presented a new model for obtaining the binding energy of lithium isotopes nucleus in a special potential. For our purpose we solved the redial of Shroedinger equation exact analytically for the Woods-Saxon potential by using SUSYQM method. The results of our calculation for the ground state binding energy show that almost in every case have a good agreement with the experimental data, particularly in \(^6\text{Li}, \(^7\text{Li}, \(^8\text{Li}, \(^9\text{Li}, \) and \(^11\text{Li}\). The best agreement may be obtain by trying to include the spin-orbit interactions in our calculations.

**REFERENCES**

Badalov, V.H., H.I. Ahmadov and S.V. Badalov, *Math-Ph*.
Kenneth, S., Krane, 1998."Introductory Nuclear Physics".