The Falkner-Skan Flow over a Wedge with Variable Parameters

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ABSTRACT

In this paper an analysis is accomplished to study and analyze the effect of variable viscosity and also heat generation on the classical Falkner-Skan flow. The main purpose is showing the results of Falkner-Skan problem with constant and variable surface temperature and variable viscosity, involving a wide range of Prandtl and exponent m (Falkner-Skan power-law parameter). In fact, the velocity and temperature distribution are graphically shown for a wide range of Prandtl numbers from 1 to 10000. The present outcomes have been made assuming a linear correlation between fluid density and temperature. The results are achieved with the direct numerical solution of the boundary layer equations.

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INTRODUCTION

In the applicable field of aerodynamics, it is interesting to survey the boundary-layer problems for two dimensional steady and incompressible laminar flow passing a wedge. The fluid flow along a static plate is a classical problem of fluid mechanics that is well known as the Blasius problem. In this case the free stream is parallel to the plate and the velocity is constant. If the wall prepares a positive angle with the free stream, the free stream accelerates along the wall and makes the Falkner-Skan flow along a wedge. Falkner and Skan (1931) described the fact that this problem admits similarity solution as happens with the Blasius problem. Hartree (1937) analyzed this problem and showed numerical results for the wall shear stress for different values of the wedge angle. Rajagopal et al. (1983) surveyed the Falkner-Skan boundary layer flow of a homogeneous incompressible second grade fluid past a wedge placed symmetrically with respect to the flow direction. Lin and Lin (1987) showed a similarity solution method for the forced convection heat transfer from isothermal or uniform-flux surfaces to fluids with different Prandtl numbers. All of these studies were done with the assumption of constant fluid properties. However, in many industrial applications this assumption is not observed and it is necessary to analyze such problems by assuming variable viscosities. It is clear that viscosities of liquids vary with temperature, as an example the viscosity of water decreases by about 24% when the temperature increases from 100°C to 50°C. One of the first attempts to solve the Falkner-Skan problem involving the variation of viscosity with temperature was done by Herwing et al. (1986). Eckert (1942) solved the Falkner-Skan flow along an isothermal wedge and showed the first wall heat transfer values. Thereafter, many solutions analyze have been achieved for different aspects of this class of boundary layer problems. When it is assumed that the fluid has constant properties then the problem is uncoupled, that is, the momentum equation has an effect on the energy equation but the energy equation has no effect on the momentum equation. However, most fluids are temperature-dependent viscosity and this property changes significantly when there is a large temperature difference. In this case the two equations are coupled and each equation influences the other. Hossain et al. (2000) surveyed the flow of a fluid with temperature dependent viscosity past a permeable wedge with uniform surface heat flux. In the field of lubricating fluids, the heat which is generated by the internal friction and also the increase in temperature have influences on the fluid viscosity and so the viscosity of the fluid can be proposed constant. The rise of temperature eventuate a local rise in the transport phenomena by decreasing viscosity throughout the momentum boundary layer and also influences the heat transfer rate at the wall. When the fluid is proposed to have constant properties, the problem modifies to uncoupled laminar boundary layer-flow and temperature changes do not influence the fluid velocity field, but the problem converts to a coupled when the thermo-physical properties are dependent to temperature. The main objective of the present paper is to show results for the classical Falkner-Skan problem with constant and variable surface temperature and variable viscosity.
variable surface temperature including both viscosity and Prandtl number variable throughout the boundary layer and covering a wide range of Prandtl numbers and exponent m (Seddeek, M., 2009).

Mathematical Formulation:
Consider the flow along a wedge which in a flowing fluid with $u$ and $v$ denoting respectively the velocity components in the $x$ and $y$ direction, where $x$ refers the coordinate along the wedge surface and $y$ refers the coordinate perpendicular to $x$ (Fig. 1). For steady, two-dimensional flow the boundary layer equations involving variable viscosity are (Pantokratoras, A., 2006):

continuity equation:
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

momentum equation:
\[
u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\rho_a} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + u_a \frac{\partial u}{\partial x}
\]

energy equation:
\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
\]

where $T$ refers the fluid temperature, $\mu$ refers the dynamic viscosity, $\alpha$ refers the thermal diffusivity, and $\rho_a$ refers the ambient fluid density. The boundary conditions has such form:

\[y = 0, \quad u = 0, \quad v = 0, \quad T = T_w\]  

\[y \rightarrow \infty, \quad u_a = u_0 x^m, \quad T = T_a\]  

where $T_w$ refers the wedge temperature, $T_a$ refers the ambient fluid temperature, $u_a$ refers the free stream velocity and $u_0$ is a constant. The exponent $m$, that called the Falkner–Skan power-law parameter, has a relation to the wedge angle $\beta$ by:

\[m = \beta / (2 - \beta)\]

The viscosity is considered to be an inverse linear function of temperature obtained by the following equation (Rahman, A., 2012):

\[\frac{1}{\mu} = \frac{1}{\mu_a} [1 + \gamma (T - T_a)]\]

Fig. 1: Geometry of problem and coordinate system.
Where $\mu$ refers the ambient fluid dynamic viscosity and $\gamma$ is a thermal property of the fluid. Eq. (7) can be rewritten as:

$$\frac{1}{\mu} = a(T - T_c)$$  \hspace{1cm} (8)

Where $a = \gamma / \mu_a$ and $T_c = T_a - 1 / \gamma$ are constant values and depend on the reference state and the thermal property of the fluid. Eqs. (1), (2) and (3) show a two-dimensional parabolic flow. Such a flow has a prevailing velocity in the stream wise coordinate which in our study is the direction along the wedge surface. The equations has a direct solution, without any transformation, using the finite difference method (Part, A., 1980). In the numerical solution of the boundary layer problems the calculation domain should be at least equal or wider than the boundary layer thickness. However, it is obvious that the boundary layer thickness increases with x. If a Cartesian grid, formed by lines of constant x and y is studied, the number of grid points within the boundary layer for small values of x, where the boundary layer is thin, is small and the computational accuracy is low. If the mesh length is decreased to have more points in the boundary layer at small x, the grid points at large x becomes excessive. Therefore, it would be proper to have a grid which complies with the actual shape of the boundary layer. In this study an expanding grid has been chose according to the following equation:

$$y_{out} = y_0 + cx$$  \hspace{1cm} (9)

where $y_{out}$ refers the outer boundary, c refers the spreading rate of the outer boundary and x refers the distance at the current step. The proper value of c has been found to be 0.15 (Pantokratoras, A., 2002). The forward step size $\Delta x$ augments in proportion to the width of the calculation domain and is near 1% of the outer boundary. In order to achieve a complete form of both the temperature and velocity profile at the same cross section used a non uniform lateral grid. $\Delta y$ refers small values near the wedge (many grid points near the wedge) and augments along y. The lateral grid cells were 300 (Pantokratoras, A., 2002). The solution procedure begins with a known distribution of velocity and temperature at the wedge edge (x = 0) and steps along the wedge. Flat velocity and temperature profiles were considered at the wedge edge. The discretized equations (2) and (3) at each downstream position are solved by applying using the tridiagonal matrix algorithm (TDMA). The cross-stream velocities $v$ were achieved from the continuity equation. As x become larger the successive velocity profiles become more and more equivalent and the same happens with temperature profiles. The solution procedure finishes at the point where the successive velocity and temperature profiles become the same. The results have grid independent treatment. The general form of momentum and energy equations are as follow:

$$\rho u \frac{\partial \varphi}{\partial x} + \rho v \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} \left( \Gamma \frac{\partial \varphi}{\partial y} \right) + s$$  \hspace{1cm} (10)

where $\Gamma$ refers the diffusion coefficient and S refers the source term. In the momentum equation the diffusion coefficient is equal to dynamic viscosity and in the energy equation equal to $k/c_p$. In the energy equation the source term, once assumed zero and once heat generation term. This differential equation is discretized directly and obtained the following algebraic linear:

$$a_p \varphi_p = a_w \varphi_w + a_n \varphi_n + a_s \varphi_s + b$$  \hspace{1cm} (11)

Where

$$a_w = \Gamma_w \frac{\Delta y}{\Delta x} + (\rho u)_w \Delta y$$  \hspace{1cm} (12)

$$a_n = \Gamma_n \frac{\Delta x}{\Delta y} - (\rho v)_n \Delta x$$  \hspace{1cm} (13)

$$a_s = \Gamma_s \frac{\Delta x}{\Delta y} + (\rho v)_s \Delta x$$  \hspace{1cm} (14)

$$b = s \Delta x \Delta y$$  \hspace{1cm} (15)

and $a_p$ defined as:

$$a_p = a_w + a_n + a_s$$  \hspace{1cm} (16)
Eq. (11) can be rewritten as follows:

\[-a_n \phi_n + a_p \phi_p - a_s \phi_s = a_w \phi_w + b\]  

\[\text{(17)}\]

The solution procedure begins with a known distribution of velocity and temperature at the plate edge and steps in the vertical direction. Flat initial velocity and temperature profiles were supposed. These profiles were used only to begin the calculations and their shape had no effect on the results which were taken far downstream of the edge. The coefficients \(a_p, a_n, a_s\) and the right-hand side term in the upper equation are assumed known in the current position because they are calculated using upstream specified values. For example, considering the discretized momentum equation in the second grid line (the first grid line is placed at the plate edge). The coefficient \(a_n\) is:

\[a_N = \frac{\mu_n}{\Delta x} - (\rho v)_n \Delta x\]  

\[\text{(18)}\]

and is obtained using the values of \(\mu, \rho\) and \(v\) from the first line. The dynamic viscosity is obtained using the Matthaus formula while the density is computed using the International Equation of State for Seawater. The temperature used for the above calculations is obtained from the initial temperature profile. \(V\) in the first line has zero value. The right-hand side term \(a_w \phi_w + b\) is also specified. The coefficient \(a_w\) is obtained from values at the first line like \(a_n\) and \(\phi_w\) refers the velocity at the first line. The quantity \(b\) which shows the source term of the momentum equation and involves the density is obtained also at the first line using the International Equation of State for Seawater and temperatures of the first line. If we suppose \(m\) points along each grid line we have \(m - 2\) equations similar to Eq. (17) from point 2 to point \(m - 1\). Point 1 is placed at the plate and point \(m\) is far from the plate where the ambient conditions predominate. Velocities \(u(1)\) and \(u(m)\) has always zero value. After considering the \(m - 2\) equations along the grid line we have a system of \(m - 2\) linear equations which has a tridiagonal form and the solution is achieved using the tridiagonal matrix algorithm (TDMA). By solving the upper system we have the values of vertical velocity in the second line. The next step is the computation of the cross-stream velocities \(v\) in the second line from the continuity Eq. (1). It should be pointed here that the cross-stream velocity at the plate is \(v(0)\) so we have always \(v(1) = v(0)\). The third step is the computation of the temperatures in the second line applying the discretized equations of the energy equation. Now the coefficient \(a_N\) has such form:

\[a_N = \frac{k_n}{c_p \Delta y} - (\rho c_p)_n \Delta x\]  

\[\text{(19)}\]

and is obtained applying the values of \(k, c_p, \rho\) and \(c_p\) from the first line. The thermal conductivity is computed from the Caldwell formula and the specific heat from the Bromley formula. The boundary conditions for the
energy equation are \( T(1) = T(0) \) and \( T(m) = T(a) \). After the computation of \( u, v \) and \( T \) in the second line the procedure is repeated at the third line taking into account as upstream values those obtained at the second line.

In the neighborhood of the leading edge the problem is elliptic and the present parabolic procedure have not valid nature. However, with \( x \) increasing the elliptic effects spoil rapidly and the problem turns from elliptic to parabolic. We applied the above parabolic procedure in the complete region taking into account that we are hedge were the problem has purely parabolic nature. The simultaneous of successive velocity and successive temperature profiles at large \( x \) is a token that the present problem admits similarity solution. This means that this problem could be treated with the classical similarity method applied for variable viscosity problems. In the classical similarity method the converted energy equation contains the Prandtl number, which is usually treated as a constant across the boundary layer. However, the Prandtl number can be a function of viscosity and as viscosity changes across the boundary layer, the Prandtl number changes, too. The supposition of constant Prandtl number in the classical similarity method transpose to unrealistic results when viscosity is a strong function of temperature (Pantokratoras, A., 2004). So we applied the direct solution procedure of Patankar which leads accurate results even for strong relationship between viscosity and temperature.

**Results And Discussions**

In this part, it has been shown a comparison between present study and Pantokratoras work. As an example, in a part of his work he showed the velocity distribution for ambient Prandtl number 1000 and different exponent \( m \) when \( \theta_r = 2 \). The objective of the present paper is to obtain results considering heat generation. As will be shown later the differences of the two works are obvious.

\[
\theta'(0) = -\frac{x}{\tau_w - T_w} \left[ \frac{\partial T}{\partial y} \right]_{y=0} \tag{20}
\]

\[
f'(0) = \frac{\theta_r - 1}{\theta_r} \frac{\mu_w}{\rho_w u_w} \left[ \frac{\partial u}{\partial y} \right]_{y=0} \tag{21}
\]

where \( \theta_r \) is the dimensionless temperature \( (T - T_w)/(T_w - T_a) \), \( \mu_w \) refers the fluid viscosity at the wedge surface and \( f_u \) refers the dimensionless stream function for which the following equation is valid:

\[
f' = \frac{u}{u_a} \tag{22}
\]

The Reynolds number is explained as:
\[ Re = \frac{u_a x}{v_a} \]  

(23)

and \( \theta_r \) is a constant value explained by:

\[ \theta_r = \frac{\tau_r - \tau_a}{\tau_w - \tau_a} = -\frac{1}{\gamma(\tau_w - \tau_a)} \]  

(24)

In Eqs. (20), (21) and (22) the prime shows differentiation with respect to similarity variable \( \eta \) explained as:

\[ \eta = \frac{x}{2 Re^{1/2}} \]  

(25)

When the wedge temperature is more than the ambient one (fluid heating), negative \( \theta_r \) relates to liquids and positive \( \theta_r \) to gases. The opposite occurs for fluid cooling. Here, negative \( \theta_r \) relates to gases and positive \( \theta_r \) to liquids. It should be noted here that when \( \theta_r \to \infty \) the fluid viscosity becomes identical to ambient viscosity, means, viscosity has a constant value in the boundary layer and we face the classical Falkner–Skan flow where the momentum equation is not influenced by the energy equation. When there is a strong relationship between viscosity and temperature (large) or the temperature difference between the plate and the ambient fluid has considerable, then \( \theta_r \to 0 \). For testing the accuracy of the present procedure, results were compared with those available in the literature. First we surveyed the affect of the boundary conditions at \( x = 0 \) on the results. If we suppose that \( m = 0 \) and \( \theta_r \) is infinite then we have the classical Blasius problem with constant viscosity. In Figs. 4 a and b and Fig. 5 a and b describe the velocity and temperature distribution across the boundary layer for ambient Prandtl number 1, \( m = 1 \) and different values of the viscosity parameter \( \theta_r \).

As matter of fact, the influence of the variable viscosity parameter \( \theta_r \) on the dimensionless velocity is showed in Figs. 4 a and b for \( \theta_r > 0 \) and \( \theta_r < 0 \). It is easy to see that the velocity in the boundary-layer augments with the increase of \( \theta_r \) when it has a positive value. The opposite influence of \( \theta_r \) (when it is negative) on the velocity is observed. It should be noted from these figures that when \( \theta_r \) has large value changes of \( \theta' \) are insignificant. The reason is when \( \theta_r \to \infty \), the dynamic viscosity of the fluid (\( \mu \)) equals the viscosity of the fluid (\( \mu_a \)) at the ambient temperatures and matches with the constant viscosity case. On the other hand, the influences of the variable viscosity parameter \( \theta_r \) on the temperature distribution in the boundary-layer are depicted in Figs.5 a and b for \( \theta_r > 0 \) and \( \theta_r < 0 \). These figures showed that the temperature distribution in the boundary layer diminish with the increase of \( \theta_r \). This figure also proved the fact that when \( \theta_r \) becomes large, changes in the temperature profiles become negligible. We see that for \( \theta_r < 0 \), the temperature in the boundary-layer drops with the increase of \( \theta_r \). Furthermore the heat generation term is added to the upper calculations and results showed in Figs. 4 b and 5 b. In the following part, (b) in all figures refers heat generation term.
Fig. 4: Velocity distribution for Pr = 1, m = 1 and different values for θr (a) without heat generation, and (b) with q=100kj.

![Velocity distribution for Pr = 1, m = 1 and different values for θr](image)

Fig. 5: Temperature distribution for Pr = 1, m = 1 and different values of the viscosity parameter θr (a) without heat generation, and (b) with q=100kj.

![Temperature distribution for Pr = 1, m = 1 and different values of the viscosity parameter θr](image)

In Fig.6 a and b the wall shear stress is shown as the functions of the viscosity parameter θr with and without heat generation term. It is observed, when θr is negative, an increase of θr causes an increase in $f''(0)$ and this increase becomes quick as θr approaches zero. For positive values of θr, $f''(0)$ increases with θr increasing. This figure also show that as the exponent m increases the wall shear stress increases.

The relation between the exponent m and the wedge angle is shown in table 1:

<table>
<thead>
<tr>
<th>M</th>
<th>β</th>
<th>Wedge angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.05</td>
<td>-0.105</td>
<td>Not physical</td>
</tr>
<tr>
<td>0.08</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/11</td>
<td>1/6</td>
<td>30°</td>
</tr>
<tr>
<td>1/5</td>
<td>1/3</td>
<td>60°</td>
</tr>
<tr>
<td>1/3</td>
<td>1/2</td>
<td>90°</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>180°</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>Not physical</td>
</tr>
</tbody>
</table>

Fig. 6: Variation of wall shear stress as function of viscosity parameter for different m, Pr=1 (a) without heat generation and (b) with q=100kj.

![Variation of wall shear stress as function of viscosity parameter for different m, Pr=1](image)
This table show in the Falkner–Skan problem the flow is accelerated when \( \beta > 0 \) and retarded when \( \beta < 0 \). Fig.7 a and b shows the changes of \( f''(0) \) as a function of the exponent \( m \) without and with heat generation term. It can be deduced from figure \( f''(0) \) increase with \( m \) increasing.

![Graph showing the variation of wall shear stress as a function of the exponent \( m \) for different ambient Prandtl numbers when \( \theta_r = -0.001 \) (a) without heat generation, and (b) with \( q=100kJ \).](image)

Fig. 7: Variation of wall shear stress as a function of the exponent \( m \) for different ambient Prandtl numbers when \( \theta_r = -0.001 \) (a) without heat generation, and (b) with \( q=100kJ \).

In Fig.8 a and b, Fig.9 a and b and Fig.10 a and b temperature distribution is depicted for positive values of the viscosity parameter \( \theta_r \) and ambient Prandtl numbers 1, 1000 and 10000 without and with heat generation term, respectively. From these figures, we understand the fact that the temperature diminishes with the increasing exponent \( m \). It is also obvious in these figures, the maximum temperature can be seen for the flow over plate which is flat. Different values of the Prandtl number in the boundary layer is exerted in these figures and reveals the fact that with increasing the Prandtl number we face temperature decreasing. In general, with raising the Pr number the wall heat transfer also raises. On the other hand, the local rate of heat transfer in a fluid with constant Prandtl number has a higher value than in a fluid with variable Prandtl number when it has positive value while the opposite outcomes are discovered for negative values of \( \theta_r \). It is interesting to point that as the ambient Prandtl number augments the temperature profiles become narrower means the width of temperature profile reduces. The exponent \( m \) has the identical treatment, means that, with increasing \( m \), the temperature profile width reduces.

In Figs.11 a and b, Figs.12 a and b and Figs.13 a and b the analogous velocity profiles for positive \( \theta_r \) without and with heat generation term has been depicted. Viscosity reduces within the boundary layer for \( \theta_r < 0 \) and augments for \( \theta_r > 0 \). In fluid mechanics we see the fact that with viscosity decreasing the velocity augments, which means the fluid can flow easier and vice versa. The influence of the \( m \) parameter on the dimensionless velocity is also depicted in these figures for the values 0, 1 and 2. The value of \( m=0 \) refers to wedge angle of zero degree, means the flat plate. It is obvious in this figure that as the wedge angle parameter augments the fluid velocity also augments. The results also display that the velocity distributions became sharper for larger values of \( m \) parameter. The exponent \( m \) can be a measurement of the pressure gradient, so the positive values of exponent \( m \) refer a negative pressure gradient. In accelerating flows, where we face positive values of \( m \), velocity distributions impact closer to the surface of the wall, so the backflow event does not happen.
Fig. 8: Temperature distribution for ambient Prandtl number 1 and different values of the exponent m when $\theta_j = 2$ (a) without heat generation, and (b) with $q=100\,kJ$.

Fig. 9: Temperature distribution for ambient Prandtl number 1000 and different values of the exponent m when $\theta_j = 2$ (a) without heat generation, and (b) with $q=100\,kJ$.

It can be mentioned here the boundary-layer thickness reduces with the augmenting values of a variable viscosity parameter when it has positive values while it augments for increasing values of a variable viscosity parameter when it has negative values. In additional, with augmenting the wedge angle parameter, the expansion of the boundary-layer thickness reduces.
Fig. 10: Temperature distribution for ambient Prandtl number 10000 and different values of the exponent m when \( \theta_x = 2 \) (a) without heat generation, and (b) with q=100kj.

Fig. 11: Velocity distribution for ambient Prandtl number 1 and different exponent m when \( \theta_x = 2 \) (a) without heat generation, and (b) with q=100kj.

Fig. 12: Velocity distribution for ambient Prandtl number 1000 and different exponent m when \( \theta_x = 2 \) (a) without heat generation, and (b) with q=100kj.

Fig. 13: Velocity distribution for ambient Prandtl number 10000 and different exponent m when \( \theta_x = 2 \) (a) without heat generation, and (b) with q=100kj.
Conclusions:
In this paper, we discuss classical Falkner-Skan flow over a wedge and give the similarity solutions for steady boundary layer flow and heat generation over a wedge in presence of temperature-dependent fluid viscosity. The effects of variable viscosity parameter, fluid Prandtl number, temperature and other parameters are analyzed and also the effect of heat generation showed in figures. The set of governing equations are solved the results are achieved with the direct numerical solution of boundary layer equations. In the following some of the important results of the effect of different parameters in the classical Falkner–Skan flow come in summary: wall heat transfer \( \theta'(0) \) and the wall shear stress \( f^*(0) \) increasing with the viscosity parameter \( \theta_v \) increasing. The increase becomes quick when \( \theta_v \) approaches zero. It is observed increasing in wall heat transfer \( \theta'(0) \) as \( \text{Pr}_a \) increases for positive and negative values of \( \theta_v \). The wall heat transfer \( \theta'(0) \) and the wall shear stress \( f^*(0) \) have an increasing treatment as the exponent \( m \) increases.

REFERENCES