Numerical method for solving Lane – Emden type equations arising in astrophysics

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ABSTRACT

In this paper we introduce a new matrix method based on shifted Jacobi operational matrix for solving Lane – Emden type equations. The Lane – Emden type equations arising in astrophysics. First we introduce the shifted Jacobi operational matrix of derivative and then by use of this operational matrix we numerically solved Lane – Emden type equations. Some illustrative examples are given to demonstrate the efficiency and validity of the proposed method.

INTRODUCTION

The Lane – Emden differential equation of the well studied classical equation of nonlinear mechanics, named after the pioneering work of Lane (J.H., 1870) and Emden (1907), the equation describes the equilibrium of nonrotating polytropic fluids in self-gravitating star. The equation has been studied extensively by physicists because of its application in astrophysics and also because of its importance in the Kinetics of combustion and the Landau-Ginzburg critical phenomena (Dixon, J.M., J.A., Tusynski, 1990; Fermi, E., 1927; Fowler, R.H., 1930; Frank-kamenetskti, D.A., 1995). For mathematicians, fascination with the Lane–Emden equation might derive partly from its nonlinearity and singular behavior at the origin. Solving the Lane–Emden equation analytically in closed form is only possible for the polytropic indices ($n = 0,1,5$). The studies of singular initial value problems modeled by second order nonlinear ordinary differential equations (ODEs) have attracted many mathematicians and physicists. One of the equations in this category is the following Lane–Emden type equations:

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x,y) = g(x), \quad \alpha, x \geq 0, \quad (1)$$

with initial conditions are

$$y(0) = a \quad \text{and} \quad y'(0) = 0. \quad (2)$$

Where the prime denotes the differentiation with respect to $x$, $\alpha$ is constant, $f(x,y)$ is a nonlinear function of $x$ and $y$. It is well known that an analytic solution of Lane–Emden type equations (1) is always possible (Davis, H.T., 1962), in the neighborhood of the singular point $x = 0$. Taking $\alpha = 2$, $f(x,y) = y^n$, $g(x) = 0$ and $a = 1$ in Eqs. (1) and (2), respectively, we get

$$y''(x) + \frac{2}{x} y'(x) + y^n = 0, \quad x \geq 0, \quad (3)$$

which has another form,

$$\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) + y^n = 0, \quad (4)$$

with the initial conditions:

$$y(0) = 1, \quad y'(0) = 0. \quad (5)$$

Classically, Eqs. (4) and (5) are known as the Lane – Emden equations. Similarly, by choosing $\alpha = 2$, $f(x,y) = e^y, g(x) = 0, a = 1$ in Eqs. (1) and (2), isothermal gas spheres equation are modeled by

$$y''(x) + \frac{2}{x} y'(x) + e^y(x) = 0, \quad x \geq 0, \quad (6)$$

with initial conditions

$$y(0) = 0, \quad y'(0) = 0. \quad (7)$$

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E-mail: maleknejad@iust.ac.ir,
The solutions of the Lane–Emden equations for a given index-$n$ are known as polytrophic index-$n$. In Eq. (3), the parameter $n$ physical significance in the range $0 \leq n \leq 5$. Eq. (3) initial conditions (5) has well known analytical solutions for $n = 0, 1, 5$ (Chandrasekhar, S., 1967), and for other values of $n$, numerical solutions is sought. The series solution can be found by perturbation techniques and Adomian decomposition methods (ADM). However, these solutions are often, convergent in restricted regions. Thus, some techniques such as Padé method are required to enlarge the convergent regions (Wazwaz, A.M., 2001).

Recently, a number of methods have been proposed to solve Eq. (1) with $\alpha = 2$, $f(x, y) = f(y)$, a function of $y$ alone and $g(x) = 0$, some recent techniques are quasilinearization method (Mandelzweig, V.B., F. Tabakin, 2001; Krives, R., V.B. Mandelzweing, 2001; Krives, R., V.B. Mandelzweing, 2008), a piecewise linearization technique (Ramos, J.I., 2003) based on the piecewise linearization of the Lane–Emden equation and the analytic solution of the coefficients ordinary differential equations, the homotopy analysis method (HAM) (Liao, S.J., 2003), and a variational approach using a semi–inverse method to obtain variational principle [15] quasilinearization and may employ the Ritz technique to obtain approximate solutions (He, J.H., 2003; He, J.H., 2003; He., J.H., 2003). Later, Singh et al. (2009), applied modified homotopy analysis method (MHAM) for the first time to obtain an approximate solution of the previous solution obtained by ADM and HPM. Yousefi (2006), has obtained the numerical solution of the Lane–Emden equation (1) by converting in to an integral and then using. Legendre wavelets for $0 \leq x \leq 1$. Hybrid functions has been also used by Marzban et al. (2008) to find out the numerical solution of (1) for some particular nonlinear case in 2008. In same year, Dehghan and Shakeri (2008) used the exponential transformation $x = e^t$ with $\alpha = 2$, $f(x, y) = f(y)$ and $g(x) = 0$ to get

$$
y(t) + \dot{y}(t) + e^{2t}f(y(t)) = 0, \quad \lim_{t \to \infty} y(t) = a, \lim_{t \to \infty} e^{-t} \dot{y}(t) = 0. \quad (9)
$$

Subject to the conditions

Where the symbol denotes differentiation with respect to $t$ and then applied variational iteration method (VIM) for the approximate solution. Some more approximate solutions to Lane–Emden equations have also been proposed by using Legendre spectral method (Adibi, H., A.M. Rismani, 2010), Chebyshev polynomial collocation method (Yang, C., J. Hou, 2010), Hermite function collocation method (Parand, K., et al., 2010), Lagrangian method (Parand, K., et al., 2010), radial basis function approximation (Parand, K., et al., 2011) and optimal homotopy asymptotic method (Iqbal, S., A. Javed, 2011). In the sameyear, Van Gorder (2011; 2012) presented the perturbation technique and delta-expansion method respectively to obtain the analytic solution of singular Lane–Emden type equation. Moreover, Bhrawy and Alofi (2012) used Jacobi–Gauss collocation method and Pandey and Kumar (2012) used Bernstein’s operational matrices for solving nonlinear Lane–Emden type equations.

In present work we used Jacobi operational matrix of derivative for solving Lane–Emden type equations. As we show implementation of this method is very easy and the accuracy of answers is high. This paper is organized as follows: Section 2 represents preliminaries; in this section we introduced shifted Jacobi polynomials, and some properties of them, especially the operational matrix of derivative. In Section 3 we implemented them on Lane–Emden equation. In Section 4, some applied models are discussed. Lane–Emden equations to show the efficiency and accuracy of the proposed method. Finally, Section 5 includes a conclusion for the paper.

2. Shifted Jacobi polynomials and their operational matrix of derivative:

2.1. Shifted Jacobi Polynomials:

The well-known Jacobi polynomials are defined on the interval $[-1,1]$ and can be generated with the aid of the following recurrence formula:

$$
P_i^{(\alpha, \beta)}(t) = \frac{(\alpha + \beta + 2i - 1)\{\alpha^2 - \beta^2 + t(\alpha + \beta + 2i)\}(\alpha + \beta + 2i - 2)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-1}^{(\alpha, \beta)}(t) - \frac{1}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-2}^{(\alpha, \beta)}(t), \quad i = 2, 3, \ldots
$$

Where

$$
P_0^{(\alpha, \beta)}(t) = 1 \quad \text{and} \quad P_1^{(\alpha, \beta)}(t) = \frac{\alpha + \beta + 2}{2} t + \frac{\alpha - \beta}{2}.
$$

In order to use these polynomials on the interval $x \in [0, L]$, Doha and et al. in [33] derived the so-called Shifted Jacobi polynomials by introducing the change of variable $t = \frac{x}{L} - 1$. Let the Shifted Jacobi polynomials $P_{Li}^{(\alpha, \beta)}(x)$ be denoted by $P_{Li}^{(\alpha, \beta)}(x)$. Then $P_{Li}^{(\alpha, \beta)}(x)$ can be generated form:

$$
P_{Li}^{(\alpha, \beta)}(x) = \frac{(\alpha + \beta + 2i - 1)\{\alpha^2 - \beta^2 + \frac{2x}{L} - 1\}(\alpha + \beta + 2i - 2)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{Li-1}^{(\alpha, \beta)}(x)
$$

subject to the conditions $\lim_{x \to \infty} x = a, \lim_{x \to \infty} e^{-x} \dot{x}(x) = 0, \quad i = 2, 3, \ldots$.
\[-(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i) \frac{P_{i,i-2}(x)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} \quad i = 2,3, \ldots \]

Where

\[ P_{i,j}(0) = 1 \text{ and } P_{i,j}(1) = \frac{\alpha + \beta + 2i}{2} \frac{2x}{L} + \frac{a - \beta}{2}. \]

The analytic form of the Shifted Jacobi polynomials \( P_{i,j}(x) \) of degree \( i \) is given by

\[ P_{i,j}(x) = \sum_{k=0}^{\infty} (-1)^{i-k} \frac{\Gamma(i + \beta + 1)\Gamma(i + k + \alpha + \beta + 1)}{\Gamma(k + \beta + 1)\Gamma(i + \alpha + \beta + 1)(i - k)!} X^k. \]

Where

\[ P_{i,j}(0) = (-1)^{i+j} \frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)!}, \quad P_{i,j}(L) = \frac{\Gamma(i+\alpha+1)}{\Gamma(\alpha+1)!}. \]

\[ \int_0^L P_{i,j}(x)P_{j,k}(x)W_L^{(\alpha,\beta)}(x)dx = h_k, \]

Where \( W_L^{(\alpha,\beta)}(x) = x^\beta(L-x)^\alpha \) and \( h_k = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \), \( i = j \).

Let \( u(x) \) be a polynomial of degree \( n \), then it may be expressed in terms of Shifted Jacobi polynomials as

\[ u(x) = \sum_{j=0}^{n} c_j P_{i,j}(x) = C^T \Phi(x), \quad (10) \]

Where the coefficients \( c_j \) are given by

\[ c_j = \frac{1}{h_j} \int_0^L W_L^{(\alpha,\beta)}(x)u(x) P_{i,j}(x)dx, \quad j = 0,1, \ldots \]

If the Shifted Jacobi coefficient vector \( C \) and the Shifted Jacobivector \( \Phi(x) \) are written as

\[ C = [c_0, c_1, \ldots, c_N], \quad (11) \]

and

\[ \Phi(x) = [P_{i,0}(x), P_{i,1}(x), \ldots, P_{i,N}(x)]^T. \]

\[ \frac{d\Phi(x)}{dx} = D^{(1)} \Phi(x), \quad (13) \]

Where \( D^{(1)} \) is the \((N+1) \times (N+1)\) operational matrix of derivative given by

\[ D^{(1)} = \begin{cases} C_1(i,j) & i > j \newline 0 & \text{ otherwise}, \end{cases} \]

Where

\[ C_1(i,j) = L^{\alpha+\beta+1}(i+\alpha+\beta+1)(i+\alpha+\beta+2),(i+j-1)! L^{\alpha+\beta+1}(i+j+\alpha+\beta+1)(i+j+\alpha+\beta+2),j-1! \times F_2 \left( \frac{-i+j+1, j+i+\alpha+\beta+2, j+i+\alpha+1}{j+i+\alpha+1, 2j+i+\alpha+2} \right). \]

For the proof, see [34], and for the general definition of a generalized hyper geometric series and special \( F_2 \), see [35], pp. 103-104, respectively.

For example, for even \( N \) we have

\[ D^{(1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\
0 & C_1(1,0) & 0 & \cdots & 0 \\
C_1(2,0) & C_1(2,1) & \cdots & 0 & 0 \\
C_1(3,0) & C_1(3,1) & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_1(N,0) & \cdots & C_1(N,N-1) & C_1(N,N-1) & 0 \end{bmatrix} \]

By using (13), it is clear that

\[ \frac{d^n\Phi(x)}{dx^n} = (D^{(1)})^n \Phi(x). \quad (14) \]

Where \( n \in \mathbb{N} \) and the superscript in \( D^{(1)} \), denotes matrix powers. Thus
\[ D^{(1)} = (D^{(1)})^n, \quad n = 1, 2, \ldots. \]

3. Application of the shifted Jacobi polynomials of derivatives:

This section presents the implementation of the proposed method for solving Lane–Emden type equations. Let consider the Lane–Emden equations of the form

\[
y''(x) + \frac{n}{x}y'(x) + f(x, y) = g(x), \quad \alpha, x \geq 0, \tag{15}
\]

With initial conditions:

\[
y(0) = a, \quad y'(0) = 0.
\]

Approximating \( y(x), f(x, y) \) and \( g(x) \) by shifted Jacobi polynomials as

\[
y(x) \approx \sum_{i=0}^{N} c_i P_i^{(a, \beta)}(x) = C^T \Phi(x), \tag{17}
\]

\[
f(x, y) \approx f(x, C^T \Phi(x)) = H^T \Phi(x),
\]

\[
g(x) \approx \sum_{i=0}^{N} c_i P_i^{(a, \beta)}(x) = G^T \Phi(x),
\]

Where the unknowns are \( C = [c_0, \ldots, c_N]^T \) and \( H = [h_0, \ldots, h_N]^T \).

Using operational matrix of differentiation Jacobi polynomial, Eq. (15) can be written as

\[
C^T D^{(2)} \Phi(x) + \frac{n}{x} C^T D^{(1)} \Phi(x) + H^T \Phi(x) \approx G^T \Phi(x). \tag{18}
\]

The initial condition (16) are given by

\[
y(0) = C^T \Phi(0) = d_0, \quad y'(0) = C^T D^{(1)} \Phi(0) = d_1.
\]

Eqs. (18) and (19) give two linear equations. Since the total unknowns for vector \( C \) in Eq. (17) is \( N + 1 \), we collocate Eq. (18) in \( (N - 2) \) points \( x_i \) in the interval \([0, 1] \) as

\[
x_p = \frac{2p - 1}{2(n + 1)}, \quad p = 1, 2, \ldots, n - 2. \tag{20}
\]

Then we will have

\[
C^T D^{(2)} \Phi(x_i) + \frac{\alpha}{x_i} C^T D^{(1)} \Phi(x_i) + H^T \Phi(x_i) \approx G^T \Phi(x_i). \tag{21}
\]

For \( i = 1, 2, \ldots, N - 2 \). Now the resulting Eq. (19) and (21) generate a system of \((N + 1)\) nonlinear equations which can be solved using Newton’s iterative method. We used the Mathematica 8 software to solve this nonlinear system.

Illustrative examples:

In this section Lane–Emden type equations have been solved using the proposed method. Some special cases on Eq. (1) have been considered to illustrate the efficiency of the proposed method.

Example 1:

Consider the following nonlinear Lane-Emden equation (Bhrawy, A.H., A.S. Alofi, 2012):

\[
y''(x) + \frac{2}{x}y'(x) + 4(2e^x + x^{3/2}) = 0, \quad 0 < x < 1,
\]

The initial conditions

\[
y(0) = 0, \quad y'(0) = 0.
\]

Which has the following analytical solution:

\[
y(x) = -2\ln(1 + x^2).
\]

We solved this problem by the discussed method in this paper and the results are tabulated in Table 1 for \( n = 5 \) and \( N = 7 \) and the graph of this example for \( N = 5 \) is shown in Fig 1.

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( n = 5, N = 5 )</th>
<th>( n = 5, N = 7 )</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.410301925045328 x 10^{-17}</td>
<td>7.454391538161295 x 10^{-17}</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.07944325742188964</td>
<td>-0.07841142391947277</td>
<td>-0.07944142630656266</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.296863327179394</td>
<td>-0.296859880797258</td>
<td>-0.29684001023654644</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.614657306844258</td>
<td>-0.614971591497328</td>
<td>-0.614969394959214</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.9854149651785259</td>
<td>-0.988118847330205</td>
<td>-0.98939248672214</td>
</tr>
<tr>
<td>1.0</td>
<td>-1.3744915708740482</td>
<td>-1.3743826029381236</td>
<td>-1.386294361199906</td>
</tr>
</tbody>
</table>
Example 2:
Consider the following nonlinear Lane-Emden equation (Pandey, R.K., N. Kumar, 2012):

\[ y''(x) + \frac{2}{x} y'(x) + y^m(x) = 0, \quad 0 < x < 1, \]

The initial conditions
\[ y(0) = 1, \quad y'(0) = 0. \]

For \( m = 5 \) has the exact solution:
\[ y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}. \]

The results of this example are tabulated in Table 2 for \( n = 7 \) and \( N = 5,8 \) also the absolute errors diagram is shown in Fig 2.

Table 2: Numerical results and exact solution of example 2 (By \( \alpha = \beta = 1/2 \)).

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( n = 7, N = 7 )</th>
<th>( n = 7, N = 8 )</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.9933992643663808</td>
<td>0.9933992666444683</td>
<td>0.993399267907028</td>
</tr>
<tr>
<td>0.2</td>
<td>0.974354709791806</td>
<td>0.9743547022920847</td>
<td>0.9743547034924463</td>
</tr>
<tr>
<td>0.4</td>
<td>0.944911876614282</td>
<td>0.9449117890456539</td>
<td>0.944911825230968</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9078430165966713</td>
<td>0.9078423781998256</td>
<td>0.90784299032037</td>
</tr>
<tr>
<td>0.8</td>
<td>0.866020127162572</td>
<td>0.8660655866509445</td>
<td>0.8660254037844386</td>
</tr>
<tr>
<td>1.0</td>
<td>0.866020127162572</td>
<td>0.8660655866509445</td>
<td>0.8660254037844386</td>
</tr>
</tbody>
</table>

Fig. 1: Graph of approximate and exact solution for Example 1 by \( N = 5 \).

Fig. 2: Graph of absolute error for \( N = 7 \) and \( N = 8 \).

Example 3:
For our last example we consider a real applied model that is coincide with the isothermal gas spheres equation (Pandey, R.K., N. Kumar, 2012):
\[ y''(x) + \frac{2}{x} y'(x) + e^{y(x)} = 0, \quad 0 < x < 1, \]

with the initial conditions:

\[ y(0) = 0, \quad y'(0) = 0. \]

We solved this example by \( n = 5 \) and \( N = 3, 7, 10 \) which the results are tabulated in Table 3.

### Table 3: Numerical results of example 3 (By \( \alpha = \beta = 1/2 \)).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( n = 5, N = 3 )</th>
<th>( n = 5, N = 7 )</th>
<th>( n = 5, N = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>(-2.06 \times 10^{-18})</td>
<td>5.49 \times 10^{-18}</td>
<td>(-2.16 \times 10^{-18})</td>
</tr>
<tr>
<td>0.1</td>
<td>(-0.0016 \times 10^{-18})</td>
<td>(-0.0016 \times 10^{-18})</td>
<td>(-0.0016 \times 10^{-18})</td>
</tr>
<tr>
<td>0.2</td>
<td>(-0.0066 \times 10^{-18})</td>
<td>(-0.0066 \times 10^{-18})</td>
<td>(-0.0066 \times 10^{-18})</td>
</tr>
<tr>
<td>0.3</td>
<td>(-0.0149 \times 10^{-18})</td>
<td>(-0.0149 \times 10^{-18})</td>
<td>(-0.0149 \times 10^{-18})</td>
</tr>
<tr>
<td>0.4</td>
<td>(-0.0265 \times 10^{-18})</td>
<td>(-0.0265 \times 10^{-18})</td>
<td>(-0.0265 \times 10^{-18})</td>
</tr>
<tr>
<td>0.5</td>
<td>(-0.0415 \times 10^{-18})</td>
<td>(-0.0415 \times 10^{-18})</td>
<td>(-0.0415 \times 10^{-18})</td>
</tr>
<tr>
<td>0.6</td>
<td>(-0.0598 \times 10^{-18})</td>
<td>(-0.0598 \times 10^{-18})</td>
<td>(-0.0598 \times 10^{-18})</td>
</tr>
<tr>
<td>0.7</td>
<td>(-0.0814 \times 10^{-18})</td>
<td>(-0.0814 \times 10^{-18})</td>
<td>(-0.0814 \times 10^{-18})</td>
</tr>
<tr>
<td>0.8</td>
<td>(-0.1063 \times 10^{-18})</td>
<td>(-0.1063 \times 10^{-18})</td>
<td>(-0.1063 \times 10^{-18})</td>
</tr>
<tr>
<td>0.9</td>
<td>(-0.1346 \times 10^{-18})</td>
<td>(-0.1346 \times 10^{-18})</td>
<td>(-0.1346 \times 10^{-18})</td>
</tr>
<tr>
<td>1.0</td>
<td>(-0.1662 \times 10^{-18})</td>
<td>(-0.1662 \times 10^{-18})</td>
<td>(-0.1662 \times 10^{-18})</td>
</tr>
</tbody>
</table>

### Conclusion:
In this research, we have presented the numerical method based on shifted-Jacobipolynomials for solving nonlinear singular Lane-Emden equation. Byuse of shifted Jacobi polynomials as basis and the operational matrix of derivative of these functions we convert such problems to a nonlinear system that can be solved by Newton’s method. The implementation of current approach is very simple, and we can execute this method on a computer speedy.

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