Journal of Applied Science and Agriculture, 9(4) April 2014, Pages: 1907-1913

AENSI Journals

Journal of Applied Science and Agriculture

ISSN 1816-9112

Journal home page: www.aensiweb.com/jasa/index.html

Numerical method for solving Lane – Emden type equationsarising in astrophysics

K. Maleknejad, E. Hashemizadeh, F. Mahmoodi

Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad University, Alborz, Iran

ARTICLE INFO

Article history:
Received 20 January 2014
Received in revised form 16
15 April 2014
Accepted 25 April November 2014
Available online 5 May 2014

Key words:
Jacobi polynomials, operational matrix of derivative,Lane – Emden type equations,astrophysics.

ABSTRACT

In this paper we introduce a new matrix method based on shifted Jacobi operational matrix for solving Lane – Emden type equations.TheLane – Emden type equationsarising in astrophysics. First we introduce the shifted Jocobi operational matrix of derivative and then by use of this operational matrix we numerically solved Lane – Emden type equations.Some illustrative examples are given to demonstrate the efficiency and validity of the proposed method.

To Cite This Article: K. Maleknejad, E. Hashemizadeh, F. Mahmoodi., Numerical method for solving Lane – Emden type equationsarising in astrophysics. ***J. Appl. Sci. & Agric.,*** *9(4): 1907-1913, 2014*

INTRODUCTION

The Lane – Emden differential equation of the well studied classical equation of nonlinear mechanics,named after the pioneering work of Lane (Lane, J.H., 1870) and Emden (1907), the equation describes the equilibrium of nonrotating poyltropic fluids in aself-gravitating star. The equation has been studied extensively by physicists because of its application in astrophysics and also because of its importance in the Kinetics of combustion and the Landau-Ginzburg critical phenomena (Dixon, J.M., J.A., Tuszynski, 1990; Fermi, E., 1927; Fowler, R.H., 1930; Frank-kamenetskii, D.A., 1995). For mathematicians, fascination with the Lane–Emden equation might derive partly from its nonlinearity and singular behavior at the origin. Solving the Lane–Emden equation analytically in closed form is only possible for the polytropic indices $n = 0,1,5$ .The studies of singular initial value problems modeled by second order nonlinear ordinary differential equations (ODEs) have attracted many mathematicians and physicists. One of the equations in this category is the following Lane – Emden type equations:

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x,y) = g(x), \qquad \alpha, x \geq 0, \tag{1}$$

With initial conditions are

$$y(0) = a \ \ and \ \ y'(0) = 0. \tag{2}$$

Where the prime denotes the differentiation with respect to x, α is constant, $f(x,y)$ is a nonlinear function of $x$and$y$. It is well known that an analytic solution of Lane – Emden type equations (1) is always possible (Davis, H.T., 1962), in the neighborhood of the singularpoint $x = 0$. Taking $\alpha = 2$, $f(x,y) = y^n$, $g(x) = 0$ and $a = 1$ in Eqs. (1) and (2), respectively, we get

$$y''(x) + \frac{2}{x} y'(x) + y^n = 0, \qquad x \geq 0, \tag{3}$$

which has another form,

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + y^n = 0 \,, \tag{4}$$

with theinitial conditions:

$$y(0) = 1 \,, \ \ y'(0) = 0 \,. \tag{5}$$

Classically, Eqs. (4) and (5) are known as the Lane – Emden equations. Similarly, by choosing $\alpha = 2$, $f(x,y) = e^y$, $g(x) = 0$, $a = 1$ in Eqs. (1) and (2), isothermal gas spheres equation are modeled by

$$y''(x) + \frac{2}{x} y'(x) + e^{y(x)} = 0, \qquad x \geq 0, \tag{6}$$

Withinitial conditions

$$y(0) = 0 \,, \ \ y'(0) = 0. \tag{7}$$

Corresponding Author: K. Maleknejad, Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad University, Alborz, Iran.
E-mail: maleknejad@iust.ac.ir,

The solutions of the Lane – Emden equations for a given index-n are known as polytrophic index-n. In Eq. (3), the parameter n physical significance in the rang$0 \leq n \leq 5$. Eq. (3) withinitial conditions (5) has well known analytical solutions for $n = 0,1,5$ (Chandrasekhar, S., 1967), and for other values of$n$, numerical solutions is sought. The series solution can be found by perturbation techniques and Adomian decomposition methods (ADM). However, these solutions are often, convergent in restricted regions. Thus, some techniques such as Pade method are required to enlarge the convergent regions (Wazwaz, A.M., 2001).

Recently, a number of methods have been proposed to solve Eq.(1) with $\alpha = 2$, $f(x, y) = f(y)$, a function of y alone and $g(x) = 0$, same recent techniques are quasilineariziationmethod (Mandelzweing, V.B., F. Tabakin, 2001; Krives, R., V.B. Mandelzweing, 2001; Krives, R., V.B. Mandelzweing, 2008), a piecewise linearization technique (Ramos, J.I., 2003) based on the piecewise linearization of the Lane – Emden equation and the analytic solution of the coefficients ordinary differential equations, the homotopy analysis method (HAM) (Liao, S.J., 2003), and a variationalapproach using a semi – inverse method to obtain variational principle [15] quasilineariziation and may employ the Ritz technique to obtain approximate solutions (He, J.H., 2003; He, J.H., 2003; He., J.H., 2003). Later, singh *et al*. (2009), applied modified homotopy analysis method (MHAM) for the first time to obtain analytical approximate solution contains the previous solution obtained by ADM and HPM. Yousefi (2006), has obtained the numerical solution of the Lane – Emden equation (1) by converting in to an integral and then using. Legender wavelets for $0 \leq x \leq 1$. Hybrid functions has been also used by Marzban *et al*. (2008) to find out the numerical solution of (1) for some particular nonlinear case in 2008.In same year, Dehghan and Shakeri (2008) used the exponential transformation $x = e^t$ with $\alpha = 2$, $f(x, y) = f(y)$ and $g(x) = 0$ to get

$$\ddot{y}(t) + \dot{y}(t) + e^{2t} f\big(y(t)\big) = 0, \tag{8}$$

Subject to the conditions

$$\lim_{t\to-\infty} y(t) = a, \qquad \lim_{t\to\infty} e^{-t}\, \dot{y}(t) = 0. \tag{9}$$

Where the symbol.denotes differentiation with respect to$t$ and then applied variational iteration method (VIM) for the approximate solution.Some more approximate solutions to Lane–Emden equationshave also been proposed by using Legendre spectral method (Adibi, H., A.M. Rismani, 2010), Chebyshev polynomial collocation method (Yang, C., J. Hou, 2010), Hermite function collocation method (Parand, K., *et al*., 2010), Lagrangian method (Parand, K., *et al*., 2010), radialbasis function approximation (Parand, K., *et al*., 2011) and optimalhomotopy asymptotic method (Iqbal, S., A. Javed, 2011). In the sameyear, Van Gorder (2011; 2011) presented the perturbation techniqueand delta-expansion method respectively to obtain the analyticsolution of singular Lane–Emden type equation. Moreover, Bhrawyand Alofi (2012) used Jacobi–Gauss collocation method and Pandey and Kumar (2012) used Bernstein's operational matrices for solvingnonlinear Lane–Emden type equations.

In present work we used Jocobi operational matrix of derivative for solving Lane – Emden type equations. As we show implementation of this method is very easy and the accuracy of answers is high. This paper is organized as follows: Section 2 represents preliminaries; in this section we introduced shiftedJacobi polynomials, and some properties of them, especiallythe operational matrix of derivative. In Section 3 we implemented them on Lane – Emden equation. In Section 4, some applied models are discussedLane – Emden equations to show the efficiency and accuracy of the proposed method. Finally, Section 5 includes a conclusion for the paper.

2. Shifted Jacobi polynomials and their operational matrix of derivative:

2.1. Shifted Jacobi Polynomials:

The well-known Jacobi polynomials are defined on the interval $[-1,1]$and can be generated with the aid of the following recurrence formula:

$$\mathrm{P}_i^{(\alpha,\beta)}(t) = \frac{(\alpha+\beta+2i-1)\{\alpha^2-\beta^2+t(\alpha+\beta+2i)(\alpha+\beta+2i-2)\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)}\mathrm{P}_{i-1}^{(\alpha,\beta)}(t) - \frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)}\mathrm{P}_{i-2}^{(\alpha,\beta)}(t), \qquad i = 2,3,\cdots$$

Where

$\mathrm{P}_0^{(\alpha,\beta)}(t) = 1$ and$\mathrm{P}_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta+2}{2}t + \frac{\alpha-\beta}{2}$.

In order to use these polynomials on the interval $x \in [0, L]$,Doha and *et al*. in [33]derived the so-called Shifted Jacobi polynomials by introducing the change of variable$t = \frac{2x}{L} - 1$. Let the ShiftedJacobi polynomials $\mathrm{P}_{L,i}^{(\alpha,\beta)}\left(\frac{2x}{L} - 1\right)$be denoted by$\mathrm{P}_{L,i}^{(\alpha,\beta)}(x)$. Then $\mathrm{P}_{L,i}^{(\alpha,\beta)}(x)$ can be generated form:

$$\mathrm{P}_{L,i}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+2i-1)\left\{\alpha^2-\beta^2+\left(\frac{2x}{L}-1\right)(\alpha+\beta+2i)(\alpha+\beta+2i-2)\right\}}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)}\mathrm{P}_{L,i-1}^{(\alpha,\beta)}(x)$$

Journal of Applied Science and Agriculture, 9(4) April 2014, Pages: 1907-1913

$$-\frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)}P_{L,i-2}^{(\alpha,\beta)}(x), \qquad i=2,3,\cdots$$

Where

$P_{L,0}^{(\alpha,\beta)}(0)=1$ and $P_{L,1}^{(\alpha,\beta)}(1)=\frac{\alpha+\beta+2}{2}\left(\frac{2x}{L}-1\right)+\frac{\alpha-\beta}{2}$.

The analytic form of the Shifted Jacobi polynomials $P_{L,i}^{(\alpha,\beta)}(x)$ of degree i isgiven by

$$P_{L,i}^{(\alpha,\beta)}(x)=\sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!\,k!\,L^k}X^k.$$

Where

$P_{L,i}^{(\alpha,\beta)}(0)=(-1)^i\frac{\Gamma(i+\beta+1)}{\Gamma(\beta+1)i!}$, $\qquad P_{L,i}^{(\alpha,\beta)}(L)=\frac{\Gamma(I+\alpha+1)}{\Gamma(\alpha+1)i!}$.

2.2. Function Approximation:

The orthogonally condition of ShiftedJacobipolynomials is

$$\int_0^L P_{L,j}^{(\alpha,\beta)}(x)P_{L,k}^{(\alpha,\beta)}(x)W_L^{(\alpha,\beta)}(x)dx=h_k,$$

Where $W_L^{(\alpha,\beta)}(x)=x^\beta(L-x)^\alpha$ and $h_k=\begin{cases}\frac{L^{\alpha+\beta+1}\Gamma(K+\alpha+1)\Gamma(K+\beta+1)}{(2K+\alpha+\beta+1)K!\Gamma(K+\beta+\alpha+1)}, & i=j,\\ 0 & i\neq j.\end{cases}$

Let $u(x)$ be a polynomial of degree n, then it may be expressed in terms of Shifted Jacobi polynomials as

$$u(x)=\sum_{j=0}^{N}c_j\,P_{L,j}^{(\alpha,\beta)}(x)=C^T\Phi(x), \tag{10}$$

Where the coefficients c_j are given by

$$c_j=\frac{1}{h_j}\int_0^L W_L^{(\alpha,\beta)}(x)u(x)\,P_{L,j}^{(\alpha,\beta)}(x)dx. \qquad j=0,1,\cdots$$

If the Shifted Jacobi coefficient vector C and the Shifted Jacobivector $\Phi(x)$ are written as

$$C^T=[c_0,c_1,\dots,c_N], \tag{11}$$

and

$$\Phi(x)=\left[P_{L,0}^{(\alpha,\beta)}(x),P_{L,1}^{(\alpha,\beta)}(x),\dots,P_{L,N}^{(\alpha,\beta)}(x)\right]^T. \tag{12}$$

2.3.Operational Matrixof Derivative:

The derivative of the vector $\Phi(x)$ can be expressed by

$$\frac{d\Phi(x)}{dx}=\boldsymbol{D}^{(1)}\,\Phi(x), \tag{13}$$

Where $\boldsymbol{D}^{(1)}$ is the $(N+1)\times(N+1)$Operational matrix of derivative given by

$$\boldsymbol{D}^{(1)}=(d_{ij})=\begin{cases}C_1(i,j) & i>j,\\ 0 & otherwise,\end{cases}$$

Where

$$C_1(i,j)=\frac{L^{\alpha+\beta}(i+\alpha+\beta+1)(i+\alpha+\beta+2)_j(j+\alpha+2)_{i-j-1}\Gamma(j+\alpha+\beta+1)}{(i-j-1)!\Gamma(2j+\alpha+\beta+1)}\times{}_3F_2\left(\begin{matrix}-i+1+j, & i+j+\alpha+\beta+2, & j+\alpha+1\\ j+\alpha+2, & 2j+\alpha+\beta+2 & ;1\end{matrix}\right).$$

For the proof, see[34], and for the general definition of a generalized hyper geometric series and special ${}_3F_2$, see [35], p. 41 and pp. 103-104, respectively.

For example, for even N we have

$$\boldsymbol{D}^{(1)}=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ C_1(1,0) & 0 & \cdots & 0 & 0\\ C_1(2,0) & C_1(2,1) & \cdots & 0 & 0\\ C_1(3,0) & C_1(3,1) & \ddots & \vdots & \vdots\\ \vdots & \vdots & & 0 & 0\\ C_1(N,0) & C_1(N,1) & \cdots & C_1(N,N-1) & 0\end{bmatrix}.$$

By using (13), it is clear that

$$\frac{d^n\Phi(x)}{dx^n}=\left(\boldsymbol{D}^{(1)}\right)^n\Phi(x). \tag{14}$$

Where $n\in N$and the superscript in$\boldsymbol{D}^{(1)}$, denotes matrix powers. Thus

Journal of Applied Science and Agriculture, 9(4) April 2014, Pages: 1907-1913

$$\boldsymbol{D}^{(1)} = \left(\boldsymbol{D}^{(1)}\right)^{n}, \qquad n = 1,2,\cdots.$$

3. Application of the shifted Jacobi polynomials of derivatives:

This section present the implementation of the proposed method for solving Lane – Emden type equations. Let consider the Lane – Emden equations of the form

$$y"(x) + \frac{\alpha}{x} y'(x) + f(x,y) = g(x), \quad \alpha, x \geq 0, \tag{15}$$

With initial conditions:

$$y(0) = a \ , \ \ y^{'}(0) = 0, \tag{16}$$

Approximating $y(x)$,$f(x,y)$ and $g(x)$ by shifted Jacobi polynomials as

$$y(x) \approx \sum_{i=0}^{N} c_i P_i^{(\alpha,\beta)}(x) = C^T \boldsymbol{\Phi}(x)\ , \tag{17}$$

$$f(x,y) \approx f\left(x, C^T \boldsymbol{\Phi}(x)\right) = H^T \boldsymbol{\Phi}(x),$$

$$g(x) \approx \sum_{i=0}^{N} c_i P_i^{(\alpha,\beta)}(x) = G^T \boldsymbol{\Phi}(x)\ ,$$

Where the unknowns are $C = [c_0, \dots, c_N]^T$ and $H = [h_0, \dots, h_N]^T$.

Using operational matrix of differentiation Jacobi polynomial, Eq. (15) can be written as

$$C^T \boldsymbol{D}^{(2)} \boldsymbol{\Phi}(x) + \frac{\alpha}{x} C^T \boldsymbol{D}^{(1)} \boldsymbol{\Phi}(x) + H^T \boldsymbol{\Phi}(x) \approx G^T \boldsymbol{\Phi}(x). \tag{18}$$

The initial condition (16) are given by

$$y(0) = C^T \boldsymbol{\Phi}(0) = d_0, \qquad y^{'}(0) = C^T \boldsymbol{D}^{(1)} \boldsymbol{\Phi}(0)$$
$$= d_1. \tag{19}$$

Eqs. (18) and (19) give two linear equations. Since the total unknowns for vector C in Eq. (17) is $(N+1)$, we collocate Eq. (18) in $(N-2)$ points x_i in the interval $[0,1]$as

$$x_p = \frac{2p-1}{2(n+1)}\ , \qquad p = 1,2,\dots,n-2\ . \tag{20}$$

Then we will have

$$C^T \boldsymbol{D}^{(2)} \boldsymbol{\Phi}(x_i) + \frac{\alpha}{x_i} C^T \boldsymbol{D}^{(1)} \boldsymbol{\Phi}(x_i) + H^T \boldsymbol{\Phi}(x_i) \approx G^T \boldsymbol{\Phi}(x_i). \tag{21}$$

For $i = 1,2,\dots,N-2$. Now the resulting Eq. (19) and (21) generate a system of $(N+1)$ nonlinear equations which can be solved using Newton's iterative method. We used the Mathematica 8 software to solve this nonlinear system.

Illustrative examples:

In this section Lane – Emden type equations have been solved using the proposed method. Some special cases on Eq. (1) have been considered to illustrate the efficiency of the proposed method.

Example 1:

Consider the following nonlinear Lane-Emden equation (Bhrawy, A.H., A.S. Alofi, 2012):

$$y"(x) + \frac{2}{x} y'(x) + 4(2e^{y} + y^{y/2}) = 0, \quad 0 < x < 1,$$

The initial conditions

$$y(0) = 0, \qquad y^{'}(0) = 0,$$

Which has the following analytical solution:

$$y(x) = -2Ln(1 + x^2).$$

We solved this problem by the discussed method in this paper and the results are tabulated in Table 1for $n = 5$ and$N = 5{,}7$ and the graph of this example for $N = 5$ is shown in Fig 1.

Table 1: Numerical results and exact solution of Example1 (By $\alpha = \beta = 1/2$).

x_i	$n = 5, N = 5$	$n = 5, N = 7$	Exact Solution
0.0	$1.4103019250445328 \times 10^{-17}$	$7.454391538161295 \times 10^{-17}$	0.0
0.2	-0.07843587274288964	-0.07844124329194727	-0.07844142630656266
0.4	-0.296833327179384	-0.2968399889772585	-0.29684001023654644
0.6	-0.6146573068442458	-0.6149715914973282	-0.6149693994959214
0.8	-0.9854149851785259	-0.9888118847330205	-0.989392483672214
1.0	-1.3744915708740482	-1.3743826029381236	-1.3862943611198906

Journal of Applied Science and Agriculture, 9(4) April 2014, Pages: 1907-1913

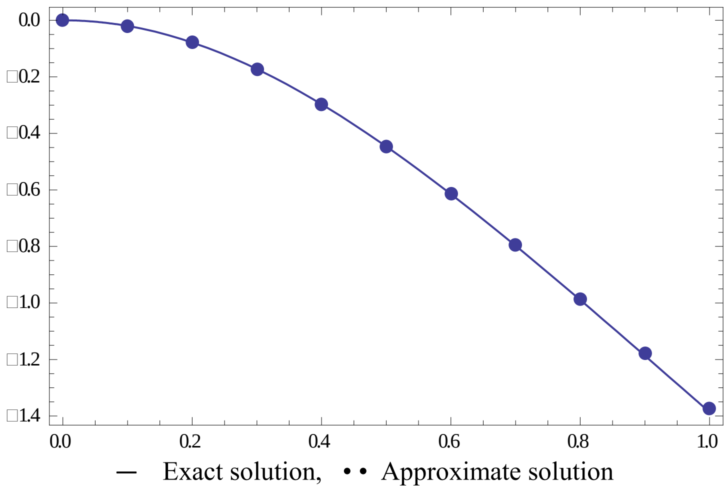

Fig. 1: Graph of approximate and exact solution for Example 1 by $N = 5$.

Example 2:

Consider the following nonlinear Lane-Emden equation (Pandey, R.K., N. Kumar, 2012):

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad 0 < x < 1,$$

The initial conditions

$$y(0) = 1, \qquad y'(0) = 0.$$

For $m = 5$ has the exact solution:

$$y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}.$$

The results of this example are tabulated in Table 2for $n = 7$ and$N = 5,8$ also the absolute errors diagram is shown in Fig 2.

Table 2: Numerical results and exact solution of example 2 (By $\alpha = \beta = 1/2$).

x_i	$n = 7, N = 7$	$n = 7, N = 8$	Exact solution
0.0	1.	1.	1.
0.2	0.9933992643663808	0.9933992666446483	0.9933992677987828
0.4	0.9743547009791806	0.9743547023820847	0.9743547036924463
0.6	0.9449111876614282	0.9449111790456539	0.944911182523068
0.8	0.9078403165966713	0.9078423781998256	0.9078412990032037
1.0	0.8660200127162572	0.8660655866509445	0.8660254037844386

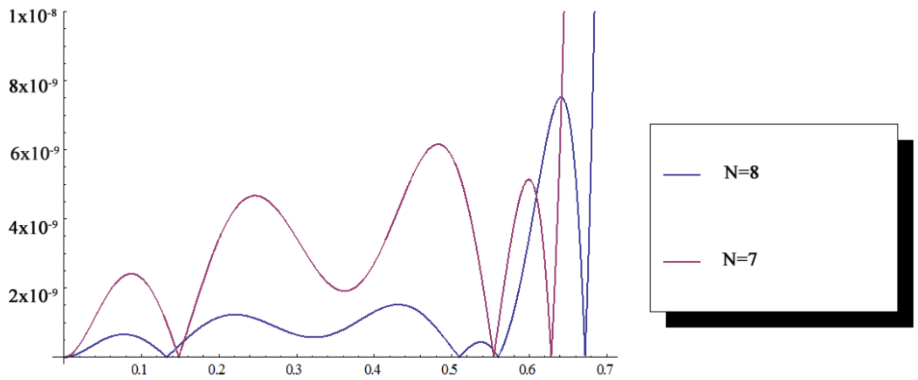

Fig. 2: Graph of absolute error for$N = 7$ and $N = 8$.

Example 3:

For our last example we consider a real applied model that is coinside with the isothermal gas spheres equation (Pandey, R.K., N. Kumar, 2012):

Journal of Applied Science and Agriculture, 9(4) April 2014, Pages: 1907-1913

$$y''(x) + \frac{2}{x}y'(x) + e^{y(x)} = 0, \quad 0 < x < 1,$$

with the initial conditions :

$$y(0) = 0, \qquad y'(0) = 0.$$

We solved this example by $n = 5$ and $N = 3,7,10$which the results are tabulated in Table 3.

Table 3: Numerical results of example 3 (By $\alpha = \beta = 1/2$).

x_i	$n = 5, N = 3$	$n = 5, N = 7$	$n = 5, N = 10$
0.0	$-2.026818324833851 \times 10^{-18}$	$5.449135329602926 \times 10^{-18}$	$-2.169539225852019 \times 10^{-18}$
0.1	-0.0016623432650043342	-0.001665833815921871	-0.001665833862071569
0.2	-0.006649373060017323	-0.006653367182420695	-0.006653367100393262
0.3	-0.014961089385038978	-0.014932883352746264	-0.014932883279792872
0.4	-0.02659749224006929	-0.02645547639840369	-0.026455476337912082
0.5	-0.04155858162510827	-0.04115395745464055	-0.041153957292674986
0.6	-0.05984435754015591	-0.05894407445695956	-0.058944074762471425
0.7	-0.08145481998521222	-0.07972601208776836	-0.07972600423521264
0.8	-0.10638996896027718	-0.10338613865227785	-0.10338605319222009
0.9	-0.1346498044653508	-0.12979896660276055	-0.12979852471375802
1.0	-0.1662343265004331	-0.1588292934302806	-0.1588276816163989

Conclusion:

In this research, we have presentedthe numerical method based on shiftedJacobipolynomials for solving nonlinear singular Lane-Emden equation. Byuse of shifted Jacobi polynomials as basis and theoperational matrix of derivative of these functions weconvert such problems to a nonlinear system that can besolved by Newton's method. The implementation of current approach is very simple, and we can execute this method on a computer speedy.

REFERENCES

Adibi, H., A.M. Rismani, 2010. Comput. Math. Appl. 60: 2126.

Bhrawy, A.H., A.S. Alofi, 2012. Commun. Nonlinear Sci. Numer. Simul., 17: 62.

Chandrasekhar, S., 1967. Introduction to study of stellar structure. Dover, New York.

Davis, H.T., 1962. Introduction to nonlinear differential and integral equation Dover, New York.

Deghhan, M., F. Shakeri, 2008. A approximate solution of a differential equation arising in astrophysics using variational method , New Astrom, 13: 53-59.

Dixon, J.M., J.A., Tuszynski, 1990. "Solutions of a generalized Emden equation and the physical significance ", Phys Rev A. 41: 416-4173.

Doha, E.H., A.H. Bhrawy, 2004. On the construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials, J. Phys. A Math. Gath. Gen., 37: 657-675.

Doha, E.H., A.H. Bhrawy, 2004. Onthe construction of recurrence relations for the expansion and connection coefficients in series of Jacobi polynomials, J. Phys.A Math. Gen., 37: 657-675.

Emden, R., 1907. GaskugelnAnwendungen der Mechan. Warmtheorie, Druck und Verlag V on B. G Teubner, Loipzing and Berlin.

Fermi, E., 1927. "Un metodo statistic per La determinaziane di alcunepriorieta dell atome", Rend. Accad. Na2, Lincei., 6: 602-607.

Fowler, R.H., 1930. " The solutions of Emdensand similar differential equations", MNRA. 91: 63-91.

Frank-kamenetskii, D.A., 1995. Diffusion and heat exchange in chemical Kinetics, Princeton University Press, Princeton.

He, J.H., 2003. Variation approach to the Six-Order boundary value problems, .Apple.Math.Comput., 143: 537-538.

He, J.H., 2003. Variation approach to the Thomas-Fermi equation. Apple.Math.Comput., 143: 533-535.

He, J.H., 2003.Variation approach to the Lane-Emden equation . Apple.Math.Comput., 143: 539-541.

He., J.H., 2003. A Lagrangian for von karman equation of the large deflection problem of thin circular plate. Apple.Math.Comput., 143: 543-549.

Iqbal, S., A. Javed, 2011. Appl. Math. Comput., 217: 7753.

Krives, R., V.B. Mandelzweing, 2001. Numerical investigation of, Quasilinearzation methodin quantum machanics, comput. Phys.commun, 138: 69-79.

Krives, R., V.B. Mandelzweing, 2008. Quasilinearzation approach to computation with singular potentials, comput. Phys.commun. doi: 10,1016j.cpc. 07.006.

Lane, J.H., 1870. “on the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment” The American Journal of Science and Arts, 5: 57-74.

Liao, S.J., 2003. A new analytic algorithm of Lane-Emden type equations. Apple.Math.Comput., 142: 1-16.

Luke, Y., 1969. The Special Functions and Their Approximations, vol. 2, Academic Press, New York.

Mandelzweing, V.B., F. Tabakin, 2001. Quasilinearzation approach to nonlinear problem sin physics with application to nonlinear ODES. Comput.Physcommun., 141: 268-289.

Marzban, H.R., H.R. Tabrizidooz, M. Razzaghi, 2008. Hybrid functions for nonlinear initial-value problems with applications to Lane-Emden type equations. J. Phys, Lett, A 372: 5883-5886.

Pandey, R.K., N. Kumar, 2012. New Astronomy, 17: 303.

Parand, K., A.R. Rezaei, A. Taghavi, 2010. ActaAstronautica, 67: 673.

Parand, K., M. Dehghan, A.R. Rezaei, S.M. Ghaderi, 2010. Comput. Phys. Commun. 181: 1096.

Parand, K., S. Abbasbandy, S. Kazem, A.R. Rezaei, 2011. PhysicaScripta, 83: 015011.

Ramos, J.I., 2003. linearization.Method in classical and quantum mechanicecomput, phys. Commun. 153: 199-208.

Singh, O.P., R.K. Pandey, V.K. Singh, 2009. An analytic algorithm for Lane-Emden equations arising in Astrophysics using MHAM, Comput, Phys. Comun., 180: 1116-1124.

Van Gorder, R.A., 2011. Celestial Mech. Dyn. Astron., 109: 137.

Van Gorder, R.A., 2011. New Astron., 16: 65.

Wazwaz, A.M., 2001. Appl. Math. Comput 118, 287.

Yang, C., J. Hou, 2010. International Conference on Computational and Information Sciences, pp: 97-100.

Yousefi, S.A., 2006. Legendre wavelet method for solving differential equations of Lane-Emden type , Appl. Math, Comput, 181: 1417-1422.