Analytical Approximate Solutions of the Systems of Non Linear Partial Differential Equations by Homotopy Perturbation Method (HPM), and Homotopy Analysis Method (HAM)

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ABSTRACT

In this article two perturbation methods, namely Homotopy Perturbation Method and Homotopy Analysis Method are applied to the non linear partial differential equations, the results have been compared. The obtain results of HPM and HAM coincide with each other. Also the relations between these two methods are also explained.

INTRODUCTION

The nonlinear phenomena played a very significant role in the field of applied Mathematics and mathematical Physics. Since in the presence of computer programming softwares, the solution of a linear equation is not a problem. But to solve nonlinear problems analytically, it is still difficult for the mathematicians. The analytical methods are fastly developing, but still have some deficiencies and shortcomings, which do not satisfy the mathematician. Like other nonlinear analytical technique, Homotopy Perturbation Method HPM and Homotopy Analysis Method (HAM) are two well known methods for obtaining analytical approximate solutions to the differential equations.

HPM was first introduced by a Chinese mathematician, J. H. He in 1999. Later this was applied by many author (Shakil, et al 2013; Wahab, et al 2013; Wahab, et al 2014; Siddiqui, et al 2014; Wahab, 2006; Siddiqui, 2011), to find the solution of variety of nonlinear problems in different fields of science. This method has the ability to solve linear and nonlinear problems. This method does not require any parameter restrictions like traditional perturbation techniques. It is also applicable to different types of equations and we have successfully applied for the system of Lotka-Volterra type of PDEs (Wahab, et al 2013), system of KdV equations (Wahab, et al 2014), Problems in fluid mechanics and heat transfer theory etc (Wahab, et al 2014; Siddiqui, et al 2014; Siddiqui, et al 2010; Wahab, 2006; Siddiqui, 2011). In most cases, the first proposed technique provides a very rapid and fast convergence. Thus a method which has the ability to solve different types of nonlinear equations is known as Homotopy Perturbation Method (HPM).

Another analytical approximation technique presented by Liao in 1992 (Shijun, 2004a). This method is based on an interesting property called homotopy, a fundamental concept of differential geometry and topology (Shijun, 1995; Shijun, 2005; Shijun, 2004a; Shijun, 2005b; Neyfeh, 2004). Homotopy Analysis Method is can be used to find the solution of nonlinear problems. Since perturbation techniques are often non valid in case of strong non linearity, but Homotopy Analysis Method (HAM) is valid in such non-linearity case (Shijun L., 2005a). By using one interesting axiom of homotopy, the problem which is non-linear can be converted into an infinite linear problems, also in this technique there is no restriction of parameter that the parameter may be small or large. If a non-linear problem has even a single solution, then through this method, there exist an infinite number of disjoint solution expressions for which the region and the rate of convergence have the
dependence on the auxiliary parameter (Shijun, 1995; Shijun, 2005; Shijun, 2004a; Shijun, 2005b; Neyfah, 2004).

In this paper we present the approximated analytical solution to some partial differential equations and systems of partial differential equations by using HPM and HAM, and then comparing the results with each other.

2 Analysis of Homotopy Perturbation Method:
Consider a Differential Equation, whose general form is given by,
\[ L(x) + N(x) = g(z), z \in \Xi \]  
subject to the boundary condition,
\[ K \left[ \frac{\partial x}{\partial t} \right] = 0, z \in \Theta \]  

In equation (1)-(2), "L" and "N" are the linear and nonlinear operators respectively. "K" is defined to be the boundary operator. Boundary of the domain "\( \Xi \)" is "\( \Theta \)". The known function is define to be the function "g(z)". Now by Homotopy Analysis Method constructing a homotopy, such that,
\[ \phi(r,q) : \Xi \times [0,1] \rightarrow \mathbb{R} \]
which satisfies,
\[ H(\phi, \sigma) = (1-\sigma)\left[ L(\phi) - L(x_0) \right] + \sigma\left[ L(\phi) + N(\phi) - g(z) \right] = 0 \]  
Equation (3), becomes
\[ H(\phi, \sigma) = L(\phi) - L(x_0) + \sigma L(x_0) + \sigma\left[ N(\phi) - g(z) \right] = 0 \]  
where "\( z \in \Theta \)" and "\( \sigma \in [0,1] \)" is known to be the embedding parameter, "\( x_0 \)" is define to be the, so called initial approximation, must satisfies the boundary condition. Now setting "\( \sigma = 0 \)" and "\( \sigma = 1 \)" in equation (4), then
\[ L(\phi) - L(x_0) = 0 \]  
\[ L(\phi) + N(\phi) - g(z) = 0 \]  
Equation (5) and equation (6), are called homotopic equations and the value of "\( \sigma \)" changing from "0" to "1" is known deformation (He, 2003). Now according to HPM, the basic supposition for the solution of equation (3)-(4), the basic assumption is that, such that,
\[ \phi = v_0 + \sigma v_1 + \sigma v_2 +..... \]  
Hence analytical approximate solution for equation (1), can be derived through Homotopy Perturbation Method (HPM) by setting "\( \sigma = 1 \)" in equation (7), which becomes,
\[ x = \lim_{q \rightarrow 0} \phi \left( x, r, \sigma \right) = v_0 + v_1 + v_2 +..... \]  
Which will be the required approximated solution of the given nonlinear problem derived by HPM.

3 Description of Homotopy Analysis Method:
HAM is very simple method, which was first discovered by Liao (Shijun, 2004a). This method was presented by means of homotopy which is a topology concept (Shijun, 2005b).
Consider a differential equation, such that
\[ N[\gamma_0(x,t)] = 0 \]  
In equation (9), "\( N \)" is non-linear operator. Now according to HAM presented by Liao (Shijun, 2004a), construct a new type of homotopy called deformation equation of zero-order, such that,
\[ (1-\sigma)\left[ \Psi(x,t;\sigma) - \gamma_0 \right] + \sigma\left[ H(\Psi(x,t;\sigma)) \right] = 0 \]  
In equation (10), "\( \gamma_0(x,t) \)" is known to be the initial approximation of the given unknown function that is, "\( \gamma \)". "\( \Psi \)" is a function, which is not known. "\( \sigma \)" is the embedding parameter, "\( H \)" and "\( H \)" are the auxiliary things, that are auxiliary parameter and auxiliary function respectively, "\( N \)" is the operator called non-linear and "\( L \)" is the auxiliary operator called linear operator. In this method it is very important that we can easily and with great freedom chose the auxiliary materials (Shijun, 1995; Shijun, 2005; Shijun, 2004a; Shijun, 2005b).
Now if "\( \sigma = 0 \)" and "\( \sigma = 1 \)" then equation (10), becomes,
\[ \Psi(x,t;0) = \gamma_0 \]  
\[ \Psi(x,t;1) = \gamma \]  
(11)
Equation (11), shows that the variation of the embedding parameter varies from "0" to "1". The variation of this kind is called deformation in the theory of topology (Shijun, 2004b). According to HAM, expending "Ψ" in a power series with respect to "σ". Such that,

\[ Ψ = γ_0 + σγ_1 + σ^2 γ_2 + ... \]  

(12)

In equation (12), \( γ_m = \frac{\partial^m Ψ}{m! \partial σ^m} \) at "σ = 0". Now if "σ"s are chosen through proper way, then the above series will be convergent at "σ = 1". Thus the solution of equation (9) becomes,

\[ γ = γ_0 + \sum_{m=1}^{∞} γ_m σ^m \]  

(13)

Which is one-solution of the original nonlinear problem. Now according to the fundamental theorem of Homotopy, consider a vector, such that,

\[ \vec{γ} = [γ_0, γ_1, ..., γ_n] \]  

(14)

Then the deformation equation of order "m" is given by

\[ L[γ_m - λ_m γ_{m-1}] = η.H.E_m(γ_{m-1}) \]  

(15)

In equation (15),

\[ λ_m = \begin{cases} 0, m ≤ 1 & \text{and } E_m(γ_{m-1}) = \frac{\partial^{m-1}NΨ(x,t,σ)}{(m-1)! \partial σ^{m-1}} \\ 1, m > 1 & \end{cases} \]  

(16)

Now if "η = -1" and "H(x,t) = 1" in equation (10), then it become homotopy constructed in HPM, which shows that the HPM is a specified case of the HAM. The genialized homotopy only not depend on the parameter "σ", but it also dependent on "η" and "H". Thus the generalized homotopy give us an opportunity for the region of convergence, which depend upon on "η" and "H". Also the generalized homotopy provide us a straight forward way for controlling and adjusting the convergence region [Shijun, 1995; Shijun, 2005; Shijun, 2004a; Shijun, 2005b]. HAM is more general and valid for non-linear and linear differential equations in many types.

4 Applications:

Example 1:

Consider a nonlinear problem describe by the DEs, such that,

\[ y'' + y = x + \ln t, \quad t > 0, \]  

(17)

with initial condition

\[ y(0,t) = \ln t, \quad y_0(0,t) = 1. \]  

(18)

Solution by HPM:

Constructing homotopy for the given problem, which satisfy the relation, such that,

\[ (1-σ)L[φ - y_0] + σ[Lφ + φ]\phi = x + \ln t, \]  

(19)

where "y_0" is the initial condition, such that,

\[ y_0 = y(0,t) = \ln t. \]  

(20)

In equation (19), "L" is the linear operator and define as, "L = \frac{\partial^2}{\partial t^2}" with "L^{-1} = \int_0^t \int_0^t \frac{1}{x} dx dt". Now assume that,

\[ φ = φ_0 + σφ_1 + .... \]  

(21)

Using equation (21) in equation (19), which implies that

\[ L[φ_0 + σφ_1 + ...] - Ly_0 + σLy_0 + σL[φ_0 + σφ_1 + ...] = x + \ln t, \]  

(22)

\[ φ(0,t,σ) = φ(0,t) + αφ(0,t) + ... = \ln t \]  

(23)
\[ \phi_0(0,t;\sigma) = \phi_i(0,t) + \sigma \phi_i(0,t) + \ldots = 1, \]  
\hfill (24)

Thus from the set of equation (22)-(23) and equation (24), we have
\[ L\phi - L\phi_0 = x + \ln t; \phi_i(0,t) = \ln t, \phi_i(0,t) = 1, \]  
\hfill (25)

\[ L\phi + L\phi_0 + L\phi_i = 0, \phi_i(0,t) = 0, \phi_i(0,t) = 0, \]  
\hfill (26)

and so on.

The solution of equation (25)-(26), gives us,
\[ \phi_i = x + \ln t. \]  
\hfill (27)

\[ \phi_i = 0. \]  
\hfill (28)

Similarly we have to find the other components that is, \( u_2, u_3, \ldots \). Now we know that
\[ \lim_{\sigma \to 0} y(x,t;\sigma) = 0, \]  
\hfill (29)

Thus using equation (27)-(28), in equation (29), we obtain,
\[ y(x,t) = x + \ln t. \]  
\hfill (30)

which agree with the exact solution of the given equation.

**Solution by HAM:**

Now solving the given problem by **HAM**, thus the deformation equation of zero-order for the given problem becomes,
\[ (1-\sigma) L[\Psi(x,t;\sigma) - \phi_0] = \sigma \eta H(x,t;\sigma), \]  
\hfill (31)

In equation (30) \( "y_0" \) is the initial approximation and define as, such that,
\[ y_0 = y(x,0) = x + \ln t. \]  
\hfill (32)

Using the deformation process the zero-order deformation equation becomes,
\[ \Psi(x,t;0) = y_0. \]  
\hfill (33)

Then \( \Psi(x,t;\sigma) = y_0 + \sum_{n=1}^{\infty} y_n \sigma^n \), where \( y_n = \frac{1}{n!} \frac{\partial^n \Psi(x,t;\sigma)}{\partial \sigma^n} \) at \( "\sigma = 0" \) exist for \( "n \geq 1" \) and converges at \( "q = 1" \). Thus the solution of the given problem becomes,
\[ y = y_0 + \sum_{n=1}^{\infty} y_n \sigma^n. \]  
\hfill (34)

Now through **HAM** the nth-order deformation equation for the given problem becomes,
\[ L^{-1}\left[y_n(x,t) - \lambda_n y_{n-1}\right] = \eta H(x,t)E_n\left(y_{n-1}\right), \]  
\hfill (35)

Applying \( \ "L^{-1}" \) on equation (33), we obtain,
\[ y_n = \lambda_n y_{n-1} + \int_0^x \eta H(x,t)E_n\left(y_{n-1}\right) dx, \]  
\hfill (36)

Here we starting with the initial approximation
\[ y_0 = x + \ln t. \]  
\hfill (37)

Now using \( "n = 1" \) and also for simplicity using \( "H(x,t) = 1" \) in equation (34), we get,
\[ y_1(x,t) = 0. \]  
\hfill (38)

Again using \( "n = 2" \) in equation (34), we obtain,
\[ y_2(x,t) = 0. \]  
\hfill (39)

Similarly by putting \( n = 3, 4, 5, \ldots \), in equation (34), we obtain \( y_3, y_4, \ldots \). Now since we know that,
\[ y = y_0 + \sum_{n=1}^{\infty} y_n \sigma^n. \]  
\hfill (40)

By substituting equation (31), equation (35) and equation (36), we get,
\[ y = x + \ln t. \]  
\hfill (41)

Which is the solution derived through **Homotopy Analysis Method (HAM)**.
Example 2:
Consider a nonlinear problem, such that,
\[ \gamma_x + g\gamma_x + \gamma = 1, \] (37)
\[ g_x - \gamma g_x - \gamma = 1, \] (38)
initial condition,
\[ \gamma_0 = \gamma(x, 0) = e^x, \quad g_0 = g(x, 0) = e^{-x}. \] (39)

Solution by HPM:
To find solution by Homotopy Perturbation Method (HPM), defining a linear operator, such that
\[ L = \frac{\partial}{\partial t}. \]
Now according to Homotopy Perturbation Method (HPM), constructing homotopy for the given problem, satisfying the relation, such that,
\[ L\left[ \phi(x, t; \sigma) - u_0 \right] + \sigma \left[ Lu_0 + \phi_x \sigma + \phi - 1 \right] = 0, \] (40)
\[ L\left[ \phi(x, t; \sigma) - v_0 \right] + \sigma \left[ Lv_0 - \phi_x \sigma - \phi - 1 \right] = 0, \] (41)
where the initial approximation are assumed to be,
\[ \gamma_0 = u_0 = \gamma(x, 0) = e^x, \] (42)
\[ g_0 = v_0 = g(x, 0) = e^{-x}. \] (43)
Now suppose that, the solution can be expressed in power of \( \sigma \), such that,
\[ \phi = \phi_0 + \sigma \phi_1 + \sigma^2 \phi_2 + ..., \] (44)
\[ \varphi = \varphi_0 + \sigma \varphi_1 + \sigma^2 \varphi_2 + ... \] (45)
Substituting equation (44)-(45), in equation (40)-(41), we get,
\[ L\left( \phi_0 + \sigma \phi_1 + \sigma^2 \phi_2 + ... - u_0 \right) + \sigma \left( Lu_0 + \left( \phi_0 \phi_0 + \sigma \phi_0 \phi_1 + \sigma^2 \phi_0 \phi_2 + \sigma \phi_1 \phi_1 \right) \right) = 0, \] (46)
\[ L\left( \varphi_0 + \sigma \varphi_1 + \sigma^2 \varphi_2 + ... - v_0 \right) + \sigma \left( Lv_0 - \left( \varphi_0 \varphi_0 + \sigma \varphi_0 \varphi_1 + \sigma^2 \varphi_0 \varphi_2 + \sigma \varphi_1 \varphi_1 \right) \right) = 0, \] (47)
\[ \phi(x, 0; \sigma) = \phi_0(x, 0) + \sigma \phi_1(x, 0) + \sigma^2 \phi_2(x, 0) + ... = e^x, \] (48)
\[ \varphi(x, 0; \sigma) = \varphi_0(x, 0) + \sigma \varphi_1(x, 0) + \sigma^2 \varphi_2(x, 0) + ... = e^{-x}, \] (49)
Thus from equation (46)-(49), we have,
\[ L\phi_0 - Lu_0 = 0, \quad \phi_0 = e^x, \] (50)
\[ L\varphi_0 - Lv_0 = 0, \quad \varphi_0 = e^{-x}, \] (51)
\[ L\phi_1 + Lu_0 - \phi_0 \phi_0 - \phi_0 - 1 = 0, \quad \phi_1 = 0, \] (52)
\[ L\varphi_1 + Lv_0 - \varphi_0 \varphi_0 - \varphi_0 - 1 = 0, \quad \varphi_1 = 0, \] (53)
\[ L\phi_2 + \phi_0 \phi_1 + \varphi_0 \phi_0 + \phi_0 = 0, \quad \phi_2 = 0, \] (54)
\[ L\varphi_2 - \varphi_0 \varphi_1 + \varphi_0 \varphi_0 + \varphi_0 = 0, \quad \varphi_2 = 0, \] (55)
and so on, thus the solution of equation (50)-(55), becomes,
\[ \phi_0 = e^x, \quad \varphi_0 = e^{-x}, \] (56)
\[ \phi_1 = -te^x, \quad \varphi_1 = te^{-x}, \] (57)
\[ \phi_2 = \frac{1}{2} t^2 e^t, \varphi_2 = \frac{1}{2} t^2 e^{-t}, \]  
\[ (58) \]

Similarly we have to find \( \phi_3, \varphi_3, \phi_4, \varphi_4, \ldots \), thus the solution of the given problem becomes

\[ \gamma_{HPM} = \left(1 - t + \frac{1}{2} t^2 + \ldots \right) e^t, \]  
\[ (59) \]

\[ g_{HPM} = \left(1 + t + \frac{1}{2} t^2 + \ldots \right) e^{-t}. \]  
\[ (60) \]

Equation (59)-(60), is the approximated solution derived by HPM, converge to the solution given by,

\[ \gamma = e^{-t}, \]  
\[ g = e^{-xt}. \]

Which is the closed form solution of the problem.

**Solution by HAM:**

Since the linear operator is already define in Homotopy Perturbation Method (HPM), that is

\[ L t \frac{\partial}{\partial t} = \eta.H.N \phi \]  
\[ (61) \]

Now through HAM, the deformation equation of zero-order, for the given nonlinear problem becomes,

\[ (1 - \sigma)L [\phi(x,t;\sigma) - \gamma_0] = \sigma.\eta.H.N \phi(x,t;\sigma), \]  
\[ (61) \]

\[ (1 - \sigma)L [\phi(x,t;\sigma) - g_0] = \sigma.\eta.H.N \phi(x,t;\sigma), \]  
\[ (62) \]

Using deformation process, we obtain,

\[ \phi(x,t;0) = \gamma_0, \]  
\[ (63) \]

\[ \phi(x,t;0) = g_0, \]  
\[ (64) \]

\[ \phi(x,t;\sigma) = \gamma \]  
\[ \phi(x,t;\sigma) = g \]  
\[ (66) \]

Then \( \phi(x,t;\sigma) = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n \sigma^n \) and \( \phi(x,t;\sigma) = g_0 + \sum_{n=0}^{\infty} g_n \sigma^n \), thus the solution of the given problem takes the form,

\[ \gamma = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n \sigma^n, \]  
\[ (67) \]

\[ g = g_0 + \sum_{n=1}^{\infty} g_n \sigma^n. \]

Now according to the fundamental theorem of Homotopy Analysis Method (HAM), the deformation equation of nth-order is given by,

\[ L [\gamma_n - \lambda_n \gamma_{n-1}] = \eta.H.E^\epsilon_n \left( \gamma_{n-1} \right), \]  
\[ (67) \]

\[ L [g_n - \lambda_n g_{n-1}] = \eta.H.E^\epsilon_n \left( g_{n-1} \right), \]  
\[ (68) \]

Now applying \( "L^{-1}\) to equation (67)-(68), we obtain,

\[ \gamma_n = \lambda_n \gamma_{n-1} + \int_0^t \eta.H.E^\epsilon_n \left( \gamma_{n-1} \right) dt, \]  
\[ (69) \]

\[ g_n = \lambda_n g_{n-1} + \int_0^t \eta.H.E^\epsilon_n \left( g_{n-1} \right) dt. \]  
\[ (70) \]
Choosing the initial approximation,
\[ \gamma_0 = e^{x}, \]  
\[ g_0 = e^{-x}. \]  

Now putting \( n = 1 \) in equation (69)-(70), and for simplicity using \( H(x, t) = 1 \) and \( \eta = -1 \), we obtain
\[ \gamma_1 = -te^{x}, \]  
\[ g_1 = te^{-x}. \]  

Similarly putting \( n = 2, 3, 4, \ldots \) in equation (69)-(70), we get more components, thus the approximated solution of the given problem by \textit{Homotopy Analysis Method (HAM)}, is as,
\[ \gamma = e^{t} \left( 1 - t + \ldots \right), \]  
\[ g = e^{-t} \left( 1 - t + \ldots \right). \]  

Which agree with the solution obtained by \textit{Homotopy Perturbation Method (HPM)}.

\textbf{Example 3:}
Consider a nonlinear system described by nonlinear partial differential equations,
\[ \frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + f = 0, \]  
\[ \frac{\partial g}{\partial t} - g \frac{\partial g}{\partial x} - \frac{\partial g}{\partial y} - g = 0, \]  
\[ \frac{\partial h}{\partial t} - h \frac{\partial h}{\partial x} - \frac{\partial h}{\partial y} - h = 0, \]  

with initial condition,
\[ f_0 = e^{xy}, \]  
\[ g_0 = e^{-xy}, \]  
\[ h_0 = e^{-xy}. \]  

Here the closed form solution and the \textit{HPM} of the given problem is given in (Shijun, 1995), that is,
\[ f_{\text{closed}} = e^{xy-t}, \]  
\[ g_{\text{closed}} = e^{-xy+t}, \]  
\[ h_{\text{closed}} = e^{-xy+yt}. \]  

\textbf{Solution by Homotopy Perturbation Method (HPM):}
\[ f_{\text{HPM}} = \left( 1 - t + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \ldots \right) e^{xy}, \]  
\[ g_{\text{HPM}} = \left( 1 + t + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \ldots \right) e^{-xy}, \]  
\[ h_{\text{HPM}} = \left( 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \ldots \right) e^{-xy+y}. \]  

\textbf{Solution by HAM:}
Now we obtaining the solution of the given problem by \textit{HAM}, to compare with the result obtained by \textit{Homotopy Perturbation Method (HPM)}, thus we have,

Defining a linear-operator \( L = \frac{\partial}{\partial t} \) with \( L^{-1} = \int_0^t (\cdot) dt \). Now according to \textit{Homotopy Analysis Method (HAM)}, the zero-order deformation equations for the given system of nonlinear PDEs becomes, such that,
\begin{align*}
(1-\sigma)L\left[\Psi_1(x, y, t; \sigma) - f_0\right] &= \sigma \eta \cdot H \cdot N \cdot \Psi_1(x, y, t; \sigma), \\
(1-\sigma)L\left[\Psi_2(x, y, t; \sigma) - g_0\right] &= \sigma \eta \cdot H \cdot N \cdot \Psi_2(x, y, t; \sigma), \\
(1-\sigma)L\left[\Psi_3(x, y, t; \sigma) - h_0\right] &= \sigma \eta \cdot H \cdot N \cdot \Psi_3(x, y, t; \sigma).
\end{align*}

In equation (84)-(86), "\(f_0, g_0\), and \(h_0\)" are the initial approximation defined as,
\begin{align*}
f_0 &= e^{xy}, \\
g_0 &= e^{xy}, \\
h_0 &= e^{-xy}.
\end{align*}

Using the process of deformation, we obtain,
\begin{align*}
\Psi_1(x, y, t; 0) &= f_0, \\
\Psi_2(x, y, t; 0) &= g_0, \\
\Psi_3(x, y, t; 0) &= h_0, \\
\Psi_1(x, y, t; 1) &= f, \\
\Psi_2(x, y, t; 1) &= g, \\
\Psi_3(x, y, t; 1) &= h.
\end{align*}

In equation (90)-(95), "\(f_0, g_0\), and \(h_0\)" are the initial approximation and "\(f, g\), and \(h\)" show the exact solution of the problem. The process of changing the value of the embedding parameter from zero to unity is known as deformation. Now it is assumed that the original problem solution can be expressed in a series of "\(\sigma\)" such that,
\begin{align*}
\Psi_1 &= f_0 + \sum_{n=1}^{\infty} f_n \sigma^n, \\
\Psi_2 &= g_0 + \sum_{n=1}^{\infty} g_n \sigma^n, \\
\Psi_3 &= h_0 + \sum_{n=1}^{\infty} h_n \sigma^n.
\end{align*}

In equation (96)-(98),
\begin{align*}
\frac{\partial^n \Psi_1}{n! \partial \sigma^n}, \frac{\partial^n \Psi_2}{n! \partial \sigma^n}, \frac{\partial^n \Psi_3}{n! \partial \sigma^n}
at \"\sigma = 0\" \text{ and exist for } \"n \geq 1\" \text{ also converges at } \"\sigma = 1\". \text{ Then the solution becomes,}
\begin{align*}
f &= f_0 + \sum_{n=1}^{\infty} f_n, \\
g &= g_0 + \sum_{n=1}^{\infty} g_n, \\
h &= h_0 + \sum_{n=1}^{\infty} h_n.
\end{align*}

Now according to the fundamental theorem of Homotopy Analysis Method (HAM), the deformation equation of nth-order becomes, such that,
\begin{align*}
L[f_n - \lambda_n f_{n-1}] &= \eta \cdot H \cdot E_+^n \left(\frac{f_{n-1}}{f_{n-2}}\right), n \geq 1, \\
L[g_n - \lambda_n g_{n-1}] &= \eta \cdot H \cdot E_+^n \left(\frac{g_{n-1}}{g_{n-2}}\right), n \geq 1, \\
L[h_n - \lambda_n h_{n-1}] &= \eta \cdot H \cdot E_+^n \left(\frac{h_{n-1}}{h_{n-2}}\right), n \geq 1.
\end{align*}
Applying \( L^{-1} \), that is the inverse operator, we get,

\[
\begin{align*}
    f_n &= \lambda_n f_{n-1} + \int_0^t \eta \, \mathrm{H} \, \mathrm{E}_n^{-} \left( \frac{f_{n-1}}{\eta} \right) \, dt, \\
    g_n &= \lambda_n g_{n-1} + \int_0^t \eta \, \mathrm{H} \, \mathrm{E}_n^{-} \left( \frac{g_{n-1}}{\eta} \right) \, dt, \\
    h_n &= \lambda_n h_{n-1} + \int_0^t \eta \, \mathrm{H} \, \mathrm{E}_n^{-} \left( \frac{h_{n-1}}{\eta} \right) \, dt.
\end{align*}
\]

(105)  
(106)  
(107)

Now choosing the initial approximation, and also for simplicity using \( \eta = 1 \),

\[
\begin{align*}
    f_0 &= e^{xy}, \\
    g_0 &= e^{xy}, \\
    h_0 &= e^{xy}.
\end{align*}
\]

(108)  
(109)  
(110)

By using \( n = 1 \) in equation (105)-(107), we obtain,

\[
\begin{align*}
    f_1 &= \eta t e^{xy}, \\
    g_1 &= -\eta t e^{xy}, \\
    h_1 &= -\eta t e^{xy}.
\end{align*}
\]

(111)  
(112)  
(113)

Now using \( n = 2 \) in equation (105)-(107), we obtain,

\[
\begin{align*}
    f_2 &= \eta t e^{xy} + \eta^2 t e^{xy} + \frac{1}{2} \eta^2 t^2 e^{xy}, \\
    g_2 &= -\eta t e^{xy} - \eta^2 t e^{xy} - \frac{1}{2} \eta^2 t^2 e^{xy}, \\
    h_2 &= -\eta t e^{xy} - \eta^2 t e^{xy} + \frac{1}{2} \eta^2 t^2 e^{xy},
\end{align*}
\]

(114)  
(115)  
(116)

Similarly by using \( n = 3, 4, 5, \ldots \) in equation (105)-(107), to obtain \( f_3, g_3, h_3 \), \( f_4, g_4, h_4 \), and so on, thus using equation (108)-(110), equation (111)-(113) and equation (114)-(116) in equation (99)-(101), we obtain,

\[
\begin{align*}
    f_{\text{HAM}} &= \left( 1 + \eta t + \eta^2 t + \frac{1}{2} \eta^2 t^2 + \ldots \right) e^{xy}, \\
    g_{\text{HAM}} &= \left( 1 - \eta t - \eta^2 t - \frac{1}{2} \eta^2 t^2 + \ldots \right) e^{xy}, \\
    h_{\text{HAM}} &= \left( 1 - \eta t - \eta^2 t + \frac{1}{2} \eta^2 t^2 + \ldots \right) e^{xy}.
\end{align*}
\]

(117)  
(118)  
(119)

Equation (117)-(119) is the solution obtained by Homotopy Analysis Method (HAM), now setting the auxiliary parameter that is \( \eta = -1 \) in equation (117)-(119), then the solution obtained by Homotopy Analysis Method (HAM), becomes,

\[
\begin{align*}
    f &= \left( 1 - t + \frac{1}{2} t^2 + \ldots \right) e^{xy}, \\
    g &= \left( 1 + t + \frac{1}{2} t^2 + \ldots \right) e^{-xy}, \\
    h &= \left( 1 + t + \frac{1}{2} t^2 + \ldots \right) e^{-xy}.
\end{align*}
\]

(120)  
(121)  
(122)

From the comparison of equation (120)-(122) and the solution obtained by HPM, it is clear that the HPM is a specified case of HAM. By setting the auxiliary parameter equal to \( \eta = -1 \) in the HAM solution, it will give us the HPM solution.
5 Conclusion: From the comparison of HAM and HPM, through some simple nonlinear problems modeled by system of partial differential equations, it is found that the HPM is a special case of HAM and by using the auxiliary parameter equal to "−1" in the solution obtaining by Homotopy Analysis Method (HAM), it give us the solution obtained by HPM. Also from the comparison of these methods Homotopy Analysis Method (HAM) can provide a convenient way for adjusting and controlling the rate of convergence, because Homotopy Analysis Method (HAM) contains a convergence control parameter. In both the methods, we are required to choose the initial guess and auxiliary linear operator as in the case of Adomian Decomposition which is a straightforward method. HAM requires the setting of the auxiliary parameter and auxiliary function, whereas HPM does not contain such parameter. By the help of these two parameters, the rate of convergence is controlled and fast convergence can be achieved.

REFERENCES


