Journal of Applied Science and Agriculture, 9(3) March 2014, Pages: 1249-1253

AENSI Journals

Journal of Applied Science and Agriculture

ISSN 1816-9112

Journal home page: www.aensiweb.com/jasa/index.html

New concepts of convexity in Banach spaces

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ARTICLE INFO

Article history:
Received 20 December 2013
Received in revised form 22 February 2014
Accepted 26 March 2014
Available online 15 April 2014

Keywords:
McShane integral
Integral convex set
Integral extreme point

ABSTRACT

In this paper, are provided a new set in Banach spaces. Using McShane integral, new concepts of convexity are introduced in Banach spaces. These results are generalized concepts that Jian-Yong Wang and Yu-Mei Ma were expressed. Convex sets and extreme points have a very important role in optimal control problems.By way of McShane integral of vector-valued functions, the McShane integral convexity of sets and the concept of McShane integral extreme points of sets are introduced in banach spaces.

To Cite This Article: Mohammad Reza Heidari Tavani., New concepts of convexity in Banach spaces. ***J. Appl. Sci. & Agric.***, *9(3): 1249-1253, 2014*

INTRODUCTION

The theory of convex sets and extreme points are a vibrant and classical field of modern mathematics with rich applications in economics and optimization. It is therefore necessary to extend this branch of mathematics. In recent years optimization problems concepts quickly expanded. For example in 2004 Yong Wang and Yu-Mei Ma introduced the concept of integral convex sets and integral extreme points and proved some theorems for these concepts (Wang and Ya, 2004). In the present paper we focus on a generalization of the notion of integral convexity. By using, instead of Bochner integral, a more general vector integral, that of McShane, we obtain some results on integral convex sets and integral-extreme points of subsets of a Banach spaces. To obtain these results some preliminary results and definitions were referred. However for more details see also (Diestel and Uhl, 1977; Gordon ,1990; Schwabik and Guoju, 2005; Wang and Ya, 2004).

Preliminaries:

First we deal with functions which take values in Banach spaces. For such functions we define the various notions of measurability and different integrals corresponding to them. Let X be a Banach space, X^* its topological dual, and $([0,1],\Sigma,\mu)$ the unit interval provided with the σ -algebra of Lebesgue measurable sets and with the Lebesgue measure.

Definition 2.1:

A function $s:[0,1]\longrightarrow X$ is called simple if there is a finite sequence $E_m\in\Sigma$, $m=1,\dots,p$ such that $E_m\cap E_n=\emptyset$ for $m\neq n$ and $[0,1]=\bigcup_{m=1}^{p} E_m$ where $s(t)=y_m\in X$ for $t\in E_m$,$m=1,\dots,p$. Denote by $J(\mu,X)=J$ the set of all simple functions defined on $[0,1]$.

Definition 2.2:

A function $f:[0,1]\longrightarrow X$ is called measurable if there exists a sequence $\{s_n\}$, $s_n\in J, n\in N$ with $\lim_{n\to\infty}\|s_n(t)-f(t)\|_X=0$ for almost all $t\in[0,1]$.

Definition 2.3:

A function $f:[0,1]\longrightarrow X$ is called weakly measurable if for each $x^*\in X^*$ the real function $x^*(f):[0,1]\longrightarrow R$ is measurable.

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Remark 2.1:

It is a simple task to define the integral of a simple function. Assume that $s : I \longrightarrow X$ is a simple function given by Definition 2.1.

Define the integral of $s: I \longrightarrow X$ as $\int_{[0,1]} s\, d\mu = \sum_{m=1}^{p} s(E_m).\mu(E_m) = \sum_{m=1}^{p} y_m.\mu(E_m)$. If $A \in \Sigma$ ands $\in J$ then define $s_A(t)=s(t).\chi_A(t)=\begin{cases} s(t) & t\in A \\ 0 & t\notin A \end{cases}$. It is easy to see that the function f_A is again simple and we set $\int_A s\, d\mu = \int_{[0,1]} s_A d\mu$.

Definition 2.4:

A measurable function $f:[0,1]\longrightarrow X$ is called Bochner integrable if there exists a sequence of simple function $\{s_n\}$ such that $\lim_{n\to\infty} \int_{[0,1]} \|s_n - f\|\, d\mu = 0$. In this case, $\int_A f d\mu$ is defined for each $A \in \Sigma$ by $\int_A f\, du = \lim_{n\to\infty} \int_A s_n\, d\mu$ where $\int_A s_n\, d\mu$ is defined in the remark 2.1.

The value $\int_A f d\mu$ is called bochner integral of function f over A and If necessary the more detailed notation $(\beta) \int_A f d\mu$ will be used for this concept of integral. The set of functions β is called the set of Bochner integrable functions.

The next theorem gives a necessary and sufficent condition for the Bochner integrability of a function $f : [0, 1] \longrightarrow X$.

Theorem 2.1:

A measurable function $f : [0, 1] \longrightarrow X$ is Bochner integrable if and only if the function $t \longrightarrow \|f(t)\|_X$ is Lebesgue integrable (i.e., $\|f(\cdot)\|_X \in L^1(\Omega)$) .

Definition:

Let a compact interval $\subset R$. A pair (t, J) of a point $t \in R$ and a compact interval $J \subset R$ is called a tagged interval, t is the tag of J .Two compact intervals $J,L \subset R$ are called non-overlapping If $J^\circ \cap L^\circ = \emptyset$ (J°, L° denote the interiors of J,L, respectively).A finite collection $\{ (t_j,I_j) ,j = 1, . . . , p\}$ of pairwise non-overlapping tagged intervals is called an M-system in I if $I_j \subset I$ for j = 1,. . . ,p. An M-system $\{ (t_j,I_j) ,j = 1, . . . , p\}$ in I is called an M- partition of the interval I if $\bigcup_{i=1}^{p} I_j = I$.

Definition 2.6:

Let a compact interval $\subset R$. Given a positive function $\delta : I \to (0, +\infty)$ called a gauge on I , a tagged interval (t, J) is said to be δ -fine if $J \subset B(t,\delta(t))$,where $B(t,\delta(t))$ is the ball in R centered at t with the radius $\delta(t)$ .M - systems or partitions are called δ -fine if all the tagged intervals (t_j,I_j),j = 1, . . . , p are δ -fine with respect to the gauge δ.

Definition 2.7:

Assume that a function $f : [0, 1] \longrightarrow X$ is given.f is McShane integrable and $J \in X$ is its McShane integral if for every $\varepsilon > 0$ there exists a gauge $\delta : [0,1] \to (0, +\infty)$ such that for every δ -fine M-partition $\{ (t_i,I_i) ,i = 1, . . . , p\}$ of [0,1] the inequality $\left\|\sum_{i=1}^{p} f(t_i).\mu(I_i) - J\right\|_X < \varepsilon$ holds .We denote $J=(M)\int_{[0,1]} fd\mu$ and M denotes the set of all McShane integrable functions.In the following we provide some properties of the McShane integral.

Theorem 2.2 (Schwabik and Guoju, 2005):

If $f : [0, 1] \longrightarrow X$ as McShane integrable then for every $\varepsilon > 0$ there is an $\eta>0$ such that if $E \subset [0,1]$ is measurable with $\mu(E) < \eta$ then $\left\|(M)\int_E f\, d\mu\right\|_X \leq \varepsilon$.

Theorem 2.3(Schwabik and Guoju, 2005):

If $f : [0, 1] \longrightarrow X$ as McShane integrable with $(M)\int_{[0,1]} fd\mu \in X$ then for every $x^* \in X^*$ the real function $x^*(f) : [0, 1] \longrightarrow R$ is McShane integrable and $(M)\int_{[0,1]} x^*(f)d\mu = x^*((M)\int_{[0,1]} fd\mu)$.

Theorem 2.4(Schwabik and Guoju, 2005):

If $f : [0, 1] \longrightarrow X$ is Bochner integrable then f is McShane integrable, i.e. $\beta \subset M$ and we have $(\beta)\int_{[0,1]} fd\mu = (M)\int_{[0,1]} fd\mu$.

Journal of Applied Science and Agriculture, 9(3) March 2014, Pages: 1249-1253

Remark 2.2:

Any McShane integrable function is Pettis integrable (see [6]) . However as shown in [1] every Pettis integrable function is Henstoch-Kurzweil-Pettis integrable .So if f is McShane integrable function then it is Henstoch-Kurzweil-Pettis integrable and tow integrals are coincide.since the mean value theorem is true for Henstoch-Kurzweil-Pettis integrable function (see [1]) then also is used to McShane integrable function .Now we express the result in the next theorem.

Theorem 2.5:

Let $f : [0, 1] \longrightarrow X$ be a McShane integrable function and $E \subset [0,1]$ is measurable with $\mu(E)>0$ then $\frac{1}{\mu(E)}(M)\int_E f d\mu \in \overline{co}\ (f(E))$ .The following concepts were introduced in (Wang and Ya, 2004):

Definition 2.7:

$A \subset X$ is said to be integral-convex (shortly, $\int$ - convex) if every $f : [0, 1] \longrightarrow X$, Bochner integrable such that $f(t) \in A$ a.e. satisfies $(\beta)\ \int_{[0,1]} f\, d\mu \in A$.

Remark 2.3.(Wang and Ya, 2004):

For any $A \subset X$, there is a smallest (closed) $\int$ - convex set containing A, namely the intersection of all (closed) $\int -$ convex sets containing A, which is called the (closed) $\int$ - convex hull of A and denoted by $co_{\int}\ A$ (respectively, $\overline{co}_{\int}\ A$).We have: $coA \subset co_{\int}\ A \subset \overline{co}_{\int}\ A \subset \overline{co_{\int}\ A}$ (*)

It follows that from (*) in Remark 2.3 any $\int$ - convex subset X is convex.

Definition 2.8:

A subset $B \subset A \in 2^X$ is called an $\int -$ extremal subset of A if for any $f : [0, 1] \longrightarrow X$ Bochner integrable one-side-continuous with $f(t) \in A$ a.e. and $(\beta)\ \int_{[0,1]} f\, d\mu \in B$ implies $f([0, 1]) \subset B$. Moreover, if B is a singleton, then it is said to be an $\int$ - extreme point of A .The set of all $\int$ - extreme point of A is called its $\int -$ extreme points set and denoted by $ext_{\int}\ A$.

Remark 2.4.(Wang and Ya, 2004):

It follows from the definition 2.8 that every $\int$ - extreme point of A is its extreme point as well, i.e., $ext_{\int}\ A \subset ext\ A$.

New concepts of $\int -$ convex ity of sets:

In this paper McShane integral was used to obtain new results instead of the Bochner one.

Definition 3.1:

$A \subset X$ is said to be McShane- integral-convex (shortly, $(M)\int$ - convex) if every measurable, McShane integrable function $f : [0, 1] \longrightarrow X$ with $f(t) \in A$ a.e. satisfies $(M)\ \int_{[0,1]} f\, d\mu \in A$.

Remark 3.1:

Since any Bochner integrable function is McShane integrable then every $(M)\int$ - convex set is $\int$ - convex.

Remark 3.2:

If the Banach space is finite dimensional, then the integability in McShane and Bochner sense coincides, therefore, by theorem 2.3 in [8], every convex subset of a finite dimensional Banach space is $(M)\int$ - convex.

Definition 3.2:

A subset $B \subset A$ is called an $(M)\int$ - extremal subset of A if for any measurable function $f : [0, 1] \longrightarrow X$ McShane integrable one-side-continuous (that is, continuous at left or at right in every point) with $f(t) \in A$ a.e. and $(M)\ \int_{[0,1]} f\, d\mu \in B$ implies $f([0, 1]) \subset B$. Moreover, if B is a singleton, then it is said to be an $\int$ - extreme point of A.The set of all $(M)\int$ - extreme point of A is called its $(M)\int$ - extreme points set and denoted by $(M)ext_{\int}\ A$.

Remark 3.3:

According to the above definition any (M) $\int$ - extreme point is $\int$ - extreme point and thus is extreme point . In other words, for any $A \subset X$ The following relationship is established (M) $ext_{\int} A \subset ext_{\int} A \subset ext\, A$.

Theorem 3.1:

Every closed $\int$ - convex subset of X is (M) $\int$ - convex.

Proof:

Let A be an closed $\int$ - convex subset of X and $f:[0,1] \longrightarrow X$ is McShane integrable function such that $f([0,1]) \subset A$ .Fix $x_0 \in A$ and $\varepsilon>0$ is arbitrary .Since $f \in M$ then according to theorem 2.2 there is an $\eta_\varepsilon>0$ such that if $E \subset [0,1]$ is measurable with $\mu(E) < \eta_\varepsilon$ then $\left\|(M)\int_E f\, d\mu\right\|_X \le \frac{\varepsilon}{2}$ .Consider the measurable sets $F_i=\{t\in[0,1]: \|f(t)\|\le i\}$. It is clear that $[0,1]=\bigcup_{i=1}^{\infty} F_i$ and so there exists $n_\varepsilon \in N$ with $\mu([0,1]\text{-}F_{n_\varepsilon}) \le \min(\eta_\varepsilon, \frac{\varepsilon}{2\|x_0\|})$. Now we define the function f_ε as follows $f_\varepsilon = f\chi_{F_{n_\varepsilon}} + x_0\chi_{([0,1]\text{-}F_{n_\varepsilon})}$ .It is clear that $f_\varepsilon([0,1]) \subset A$ and also according to theorem 2.1 we will gain $f_\varepsilon \in \beta$ .since A is an $\int$ - convex subset of X then According to Definition 2.7 and theorem 2.4 we have $(\beta)\int_{[0,1]} f_\varepsilon\, d\mu = (M)\int_{[0,1]} f_\varepsilon\, d\mu \in A$.

Moreover ,$\left\|(M)\int_{[0,1]} f\, d\mu - (M)\int_{[0,1]} f_\varepsilon\, d\mu\right\| \le \sup_{E\subseteq[0,1]\text{-}F_{n_\varepsilon}} \left\|(M)\int_E f\, d\mu\right\| + \left\|\int_{[0,1]\text{-}F_{n_\varepsilon}} x_0\, d\mu\right\| \le \varepsilon$. Hence $(M)\int_{[0,1]} f\, d\mu \in \bar{A} = A$ (A is a closed set) and proof is complete.

Remark 3.4:

From theorem 2.1 in [8] is obtained any closed convex set is (M) $\int$ - convex.

Theorem 3.2:

Every open convex subset of X is (M) $\int$ - convex .

Proof:

Let $A \subset X$ be a nonempty open convex set and and $f:[0,1] \longrightarrow X$ is measurable , McShane integrable function with $f([0,1]) \subset A$ .Replacing absolute values by norms throughout the usual proof of Lusin's theorem ,we can generalize that result to the banach valued case.There exists a compact subset $F \in \Sigma \subset [0,1]$ with $\mu(F)>0$ such that f is continuous on F and hence f(F) is a compact subset of A .For every $\in F$, it follows from the local convexity of X that there exists some convex neighborhood $V(f(t))$ of $f(t)$ such that $\overline{V(f(t))} \subset A$,then we obtain an open covering of f(F) . The compactness of f(F) implies that there exists a finite subcovering .This gives us a finite covering of F with sets of the form $f^{-1}(V(f(t))) \cap F$, i.e. $F \subseteq \bigcup_{i=1}^{k} f^{-1}(V(f(t_i))) \cap F$ .As $\mu(F)>0$,there exists an element of this covernig ,$F_{i_0} = f^{-1}(V(f(t_{i_0}))) \cap F$, with $\mu(F_{i_0})>0$. By the theorem 2.5 we have $\frac{1}{\mu(F_{i_0})}(M)\int_{F_{i_0}} fd\mu \in \overline{co}\,(f(F_{i_0})) \subset \overline{V(f(t_{i_0}))} \subset A$ .If $\mu(F_{i_0})$ =1 then proof is complete .Otherwise we have $\frac{1}{\mu([0,1]\text{-}F_{i_0})}(M)\int_{[0,1]\text{-}F_{i_0}} fd\mu \in \overline{co}\,(f([0,1]\text{-}F_{i_0})) \subset \bar{A}$ .Then it follows from the convexity of A and the interior point theorem that

$$(M)\int_{[0,1]} fd\mu = \mu(F_{i_0})\left(\frac{1}{\mu(F_{i_0})}(M)\int_{F_{i_0}} fd\mu\right) + \mu([0,1]\text{-}F_{i_0})\left(\frac{1}{\mu([0,1]\text{-}F_{i_0})}(M)\int_{[0,1]\text{-}F_{i_0}} fd\mu\right) \in A$$

Theorem 3.3:

Let $A \subset X$ be a compact set .Then (M) $ext_{\int} A \neq \emptyset$.

Proof:

Let F be the family of all compact (M) $\int$ - extremal subsets of A .Then $F \neq \emptyset$ (it contains A) and it is ordered by the relation "$\supset$" (i.e., for $C_1, C_2 \in F, C_1 \prec C_2 \Leftrightarrow C_1 \supset C_2$) .Every totally ordered subset has an upper bound (its intersection upper bound) ,then by Zorn's lemma there is a maximal element $E \in F$ .It is sufficient to prove E is a singleton. Suppose that we can find $x_1, x_2 \in E$ and $x_1 \neq x_2$.There exists $x^* \in X^*$ such that $x^*(x_1) \neq x^*(x_2)$ .Thus the $E_1 = \{x\in E: x^*(x) = \min\{x^*(x): x\in E\}\}$ is a compact proper subset of E .We claim that E_1 is (M) $\int$ - extremal in E ,thus it is an (M) $\int$ - extremal subsets of A ,this contradicts the maximality of E .If E_1 were not an (M) $\int$ - extremal subsets of E ,then there exists a measurable McShaane integrable, one-side continuous function f on [0,1] such that $f([0,1]) \subset E$ and $(M)\int_{[0,1]} f\, d\mu \in E_1$ and $t_0 \in [0,1]$ such that $f(t_0) \notin E_1$,i.e., $x^*(f(t_0)) > \min\{x^*(x): x\in E\}$.So there is $\varepsilon>0$ with $x^*(f(t_0)) > \min\{x^*(x): x\in E\} + \varepsilon$ .The one-side continuity of f imply that there is a $\delta>0$ such that for every $t \in [t_0, t_0+\delta] \subset [0,1]$ we have $x^*(f(t)) > \min\{x^*(x): x\in E\} + \varepsilon$ .Now, using theorem 2.4 we have

$$x^*\left((M)\int_{[0,1]} fd\mu\right)=(M)\int_{[t_0,t_0+\delta]} x^*fd\mu + (M)\int_{[0,1]-[t_0,t_0+\delta]} x^*fd\mu > \min\{x^*(x):x\in E\}$$

Which is a contradiction.

Theorem 3.4:

Let $A \subset X$ be a nonempty compact convex set .Then $(M)\mathrm{ext}_{\int}\ A=\mathrm{ext}\ A$.

Proof:

From theorem 3.3 we have $(M)\ \mathrm{ext}_{\int}\ A\neq\emptyset$ .However from remark 3.3 we have $(M)\ \mathrm{ext}_{\int}\ A\subset\mathrm{ext}\ A$. So It is sufficient to prove $\mathrm{ext}\ A\subset(M)\ \mathrm{ext}_{\int}\ A$ .Consider $x_0\in\mathrm{ext}\ A$ and f be a measurable ,McShane integrable one-side-continuous function with $f([0,1])\subset A$ and $(M)\int_{[0,1]} fd\mu =x_0$. We will prove $f([0,1])=\{x_0\}$ .Suppose that there is $t_0\in[0,1]$ such that $f(t_0)\neq x_0$ .There exists an open convex neighborhood $V(f(t_0))$ of $f(t_0)$ such that $x_0\notin\overline{V(f(t_0))}$ .Since f is one-side-continuous, there exists $\delta>0$ such that $f([t_0,t_0+\delta])\subset V(f(t_0))$. Theorem 2.5 implies $\frac{1}{\delta}(M)\int_{[t_0,t_0+\delta]} fd\mu \in\overline{co}\ (f([t_0,t_0+\delta]))\subset\overline{V(f(t_0))}\cap A$ and $\frac{1}{1-\delta}(M)\int_{[0,1]-[t_0,t_0+\delta]} fd\mu \in A$.

Now we have $x_0=\delta\left(\frac{1}{\delta}(M)\int_{[t_0,t_0+\delta]} fd\mu\right)+(1-\delta)\left(\frac{1}{1-\delta}(M)\int_{[0,1]-[t_0,t_0+\delta]} fd\mu\right)$.

But $x_0\in\mathrm{ext}\ A$, hence $x_0=\frac{1}{\delta}(M)\int_{[t_0,t_0+\delta]} fd\mu \in\overline{V(f(t_0))}\cap A$ and so $x_0\in\overline{V(f(t_0))}$, this contradicts the fact of $x_0\notin\overline{V(f(t_0))}$.

ACKNOWLEDGMENT

The authors would like to thanks Ramhormoz Branch, Islamic Azad university , Ramhormoz, Iran, for its kindly cooperation due to this research work.

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