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A Fifth - order Family of Modified Newton Methods

[1]Malakeh Shirmahd, [2]Mehrab Esmaeili

[1]*Department of Mathematics, Islamic Azad University, Shahrekord Branch, Shahrekord, Iran*
[2]*Department of Mathematics, Islamic Azad University, Shahrekord Branch, Shahrekord, Iran*

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ABSTRACT

Newton-Raphson method, which is used for finding a simple zero α of a nonlinear equationf(x)=o , plays an important role in engineering. Therefore, addition of a correction term to every other Newton-Raphson iteration provides a fifth-order method for finding simple zeros of nonlinear functions. A two-parameter family of such methods is developed,each family member requires the given function and its derivative to be evaluated at two points per step.

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INTRODUCTION

Nonlinear equations have an important role in scientific, economic and social problems. Therefore, solving nonlinear equations or at least obtaining a precise answer is very important. For example, in the theory of diffraction of light the roots of the equation $x - tan\,x = 0$ should be calculated. Also, in the calculation of planetary orbits, the nonlinear equation $x - asin\,x = 0$should be solved which the minimum error.In the proposed iterative methods,convergenceorder of the Newton Raphson methodhas increased considerably.They all start from an initial starting point and in the first step using the Newton Raphson method approximation of the root of a function is calculated. Then, in the second step $x_{n+1}$is calculated as the rate of convergence is much better.In these methods, at each stage the value of the derivative is replaced by approximate value of the derivative of the function. By exerting changes in the iterative process of the Newton Raphson method,the order of convergence can be promoted.For example, by replacing the approximation to the derivative function, without directly calculate the derivative of the function at each stage, better approximation of the roots have gained or in fact order of convergence is improved.Another method is presented in two steps. In the first step, the initial point is improved by Newton Raphson method and in the second step its value is used simultaneously which would increase the order of convergence. Here, theway in which convergence has been improved from the second to the fifth order is described.

Method of making a two-step iterative process

Supposean error is defined as following:

$\varepsilon_n = (x_n - \alpha)$

If the sequence is convergent of order p, as shown below ε_{n+1} can be written as a power series in terms of the ε_n

$\varepsilon_{n+1} = K\,\varepsilon_n^p + O\left(\varepsilon_n^{p+1}\right)\varepsilon_n \to 0$

Where $\varepsilon_n$is equal to $(x_n - \alpha)$and K is a function only of the derivatives of the function f at α

3-1Theorem:

A family which contains two Newton Raphson steps is defined as follows:

$$(N)\begin{cases} w_n = x_n - \dfrac{f_n}{f_n'} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)} \end{cases} \quad (1-3)$$

Which order of convergence four, and error equation is as follows:

$$\varepsilon_{n+1} = \frac{1}{24}(3F_2^3)\varepsilon_n^4 + \frac{1}{12}(2F_2^2F_3 - 3F_2^4)\varepsilon_n^5 + \cdots \qquad (2-3)$$

Corresponding Author: Malakeh Shirmahd, Department of Mathematics, Islamic Azad University, Shahrekord Branch, Shahrekord, Iran
E-mail: m.parvanehzadeh@ksc.ir

Proof:
To obtain the error equation,the Taylor expansion of the function $f(x_n)$ around the root α has been written

$$f(x_n)=\sum_{i=0}^{\infty}f^{(i)}(\alpha)\frac{(x_n-\alpha)^i}{i!}= f(\alpha)+f^{'}(\alpha)\frac{(x_n-\alpha)}{1!}+f^{''}(\alpha)\frac{(x_n-\alpha)^2}{2!}$$
$$+f^{(3)}(\alpha)\frac{(x_n-\alpha)^3}{3!}+f^{(4)}(\alpha)\frac{(x_n-\alpha)^4}{4!}+\cdots$$

Suppose that the error at each step is defined as $\varepsilon_n=(x_n-\alpha)$. Therefore, $f_n$can be written in the following form:

$$f_n=\sum_{i=0}^{\infty}f^{(i)}(\alpha)\frac{\varepsilon_n^i}{i!}= f(\alpha)+f^{'}(\alpha)\frac{\varepsilon_n}{1!}+f^{''}(\alpha)\frac{\varepsilon_n^2}{2!}+f^{'''}(\alpha)\frac{\varepsilon_n^3}{3!}+f^{(4)}(\alpha)\frac{\varepsilon_n^4}{4!}+$$

Since α is a root of the function $f(x)$,we have$f(\alpha)=0$
Thus:

$$f_n=f^{'}(\alpha)\frac{\varepsilon_n}{1!}+f^{''}(\alpha)\frac{\varepsilon_n^2}{2!}+f^{'''}(\alpha)\frac{\varepsilon_n^3}{3!}+f^{(4)}(\alpha)\frac{\varepsilon_n^4}{4!}+\cdots$$

Substitution of the value of the factorial intodenominators gives:

$$f_n=f^{'}(\alpha)\frac{\varepsilon_n}{1}+f^{''}(\alpha)\frac{\varepsilon_n^2}{2}+f^{'''}(\alpha)\frac{\varepsilon_n^3}{6}+f^{(4)}(\alpha)\frac{\varepsilon_n^4}{24}+\cdots$$

With the factorization of$f^{'}(\alpha)$ will have:

$$f_n=f^{'}(\alpha)(\varepsilon_n+\frac{\frac{f^{''}(\alpha)}{f^{'}(\alpha)}}{2}\varepsilon_n^2+\frac{\frac{f^{(3)}(\alpha)}{f^{'}(\alpha)}}{6}\varepsilon_n^3+\frac{\frac{f^{(4)}(\alpha)}{f^{'}(\alpha)}}{24}\varepsilon_n^4+\cdots)$$

According to the contract,if($\frac{f^{(i)}(\alpha)}{f^{'}(\alpha)}$) in the above equation replace with F_i we have:

$$f_n=f^{'}(\alpha)\left(\varepsilon_n+\frac{F_2}{2}\varepsilon_n^2+\frac{F_3}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\frac{F_5}{120}\varepsilon_n^5+\cdots\right)(3-3)$$

The $f_n^{'}$ can be obtained by the derivative of Equation(3-3) as follows:

$$f_n^{'}=f^{'}(\alpha)\left(1+2\frac{F_2}{2}\varepsilon_n+3\frac{F_3}{6}\varepsilon_n^2+4\frac{F_4}{24}\varepsilon_n^3+5\frac{F_4}{120}\varepsilon_n^4+\cdots\right)$$

After simplifying the coefficients, the equation is as follows:

$$f_n^{'}=f^{'}(\alpha)\left(1+F_2\varepsilon_n+\frac{F_3}{2}\varepsilon_n^2+\frac{F_4}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\cdots\right)(4-3)$$

On the other hand, the Taylor expansion of the function $f(w_n)$ around the root α can be summarized as follows:

$$f(w_n)=\sum_{i=0}^{\infty}f^{(i)}(\alpha)\frac{(\alpha-w_n)^i}{i!}= f(\alpha)+f^{'}(\alpha)\frac{(\alpha-w_n)}{1!}+f^{''}(\alpha)\frac{(\alpha-w_n)^2}{2!}$$
$$+f^{'''}(\alpha)\frac{(\alpha-w_n)^3}{3!}+f^{(4)}(\alpha)\frac{(\alpha-w_n)^4}{4!}+\cdots$$

Since α is a root of the function f,we have$f(\alpha)=0$. With the factorization of$f^{'}(\alpha)$ will have:

$$f(w_n)=f^{'}(\alpha)\left(\frac{\varepsilon\big((w_n)\big)}{1}+F_2\frac{\big(\varepsilon(w_n)\big)^2}{2}+F_3\frac{\big(\varepsilon(w_n)\big)^3}{6}+F_4\frac{\big(\varepsilon(w_n)\big)^4}{24}+\cdots\right)$$

$$f(w_n)=f^{'}(\alpha)\left(\frac{\varepsilon(w_n)}{1}+F_2\frac{(\varepsilon(w_n))^2}{2}+F_3\frac{((\varepsilon(w_n))^3}{6}+F_4\frac{\big(\varepsilon(w_n)\big)^4}{24}+\cdots\right)(5-3)$$

Also

$$\varepsilon(w_n)=(w_n-x_n)-\frac{f_n}{f_n^{'}}=\varepsilon_n-\frac{f_n}{f_n^{'}}$$
$$=\varepsilon_n-\frac{f^{'}(\alpha)(\varepsilon_n+\frac{F_2}{2}\varepsilon_n^2+\frac{F_3}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\frac{F_5}{120}\varepsilon_n^5+\cdots)}{f^{'}(\alpha)(1+F_2\varepsilon_n+\frac{F_3}{2}\varepsilon_n^2+\frac{F_4}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\cdots)}$$

$$=\frac{\varepsilon_n\left(1+F_2\varepsilon_n+\frac{F_3}{2}\varepsilon_n^2+\frac{F_4}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\cdots\right)-\left(\varepsilon_n+\frac{F_2}{2}\varepsilon_n^2+\frac{F_3}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\frac{F_5}{120}\varepsilon_n^5+\cdots\right)}{\left(1+F_2\varepsilon_n+\frac{F_3}{2}\varepsilon_n^2+\frac{F_4}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\cdots\right)}$$

$$=\frac{\left(\varepsilon_n+F_2\varepsilon_n^2+\frac{F_3}{2}\varepsilon_n^3+\frac{F_4}{6}\varepsilon_n^4+\frac{F_4}{24}\varepsilon_n^5+\cdots\right)-\left(\varepsilon_n+\frac{F_2}{2}\varepsilon_n^2+\frac{F_3}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\frac{F_5}{120}\varepsilon_n^5+\cdots\right)}{\left(1+F_2\varepsilon_n+\frac{F_3}{2}\varepsilon_n^2+\frac{F_4}{6}\varepsilon_n^3+\frac{F_4}{24}\varepsilon_n^4+\cdots\right)}$$

$$=\frac{F_2}{2}{\varepsilon_n}^2+\frac{1}{6}(2F_3-3F_2^2)\varepsilon_n^3+\frac{1}{24}(-14F_2F_3+12F_2^3+3F_4)\varepsilon_n^4 + \frac{1}{120}(-20F_3^2+120F_2^2F_3-25F_2F_4-90F_2^4+4F_5)\varepsilon_n^5+\cdots$$

Now, substitution of the $\varepsilon(w_n)$into the equation (3-5)gives:

$$f(w_n)=f'(\alpha)\left(\frac{F_2}{2}\varepsilon_n^2+\frac{1}{6}(2F_3-3F_2^2)\varepsilon_n^3+\frac{1}{24}(-14F_2F_3+15F_2^3+3F_4)\varepsilon_n^4+ +\frac{1}{120}(-20F_3^2+120F_2^2F_3-25F_2F_4-90F_2^4+4F_5)\varepsilon_n^5+\cdots\right)\quad(6-3)$$

If we derivative of $f(w_n)$ with respect to x_n in equation (6-3) (considering$\varepsilon_n=x_n-\alpha$), after simplifying the coefficients, the equation is as follows:

$$f'(w_n)=f'(\alpha)(1+\frac{F_2^2}{2}\varepsilon_n^2+\frac{1}{6}(2F_2F_3-3F_2^3)\varepsilon_n^3 +\frac{1}{24}(-11F_2^2F_3+12F_2^4+3F_2F_4)\varepsilon_n^4+\cdots)\quad(7-3)$$

Expression of $x_{n+1}-\alpha$ can also be written in the following form:

$$x_{n+1}-\alpha=w_n-\alpha-\frac{f(w_n)}{f'(w_n)}$$

With replacement of Eq.(3-6) and Eq.(3-7) will have:

$$\varepsilon_{n+1}=\frac{1}{24}(3F_2^3)\varepsilon_n^4+\frac{1}{12}(2F_2^2F_3-3F_2^4)\varepsilon_n^5+\cdots\quad(8-3)$$

Therefore, the proof is complete.

3-1-1 improving the order of convergence:

Consider the following single-parameter family ofModifiedNewton Methodsof fourth order:

$$(M_\beta)\begin{cases} w_n=x_n-\dfrac{f_n}{f_n'} \\ x_{n+1}=w_n-\dfrac{f(w_n)}{f_n'}\dfrac{f_n+\beta f(w_n)}{f_n+(\beta-2)f(w_n)}\end{cases}\quad(9-3)$$

Single-parameter methods have the order of convergence four and the error equation as follows:

$$\varepsilon_{n+1}=\frac{1}{24}(-2F_2F_3+(3+6\beta)F_2^3)\varepsilon_n^4+\frac{1}{72}(-4F_3^2+(24+36\beta)F_2^2F_3 -3F_2F_4-(18+54\beta+9\beta^2)F_2^4)\varepsilon_n^5+\cdots.\quad(10-3)$$

In this section, our main goal is to use a linear combination of a family of methods proposed to improve the convergence order. For this purpose, for each two-parameters of β_1 and β_2 consider a linear combination of each two members of this family, namely$M_{\beta_1}$and $M_{\beta_2}$and a two-stage Newton (N) method listed above as follows: $(F)=a(N)+b(M_{\beta_1})+c(M_{\beta_2})$.

In this case, the first equation in (F) will be as follows:

$$a(w_n)+b(w_n)+c(w_n)=a\left(x_n-\frac{f_n}{f_n'}\right)+b\left(x_n-\frac{f_n}{f_n'}\right)+c\left(x_n-\frac{f_n}{f_n'}\right)$$

And the second equation in (F) is converted to as follows:

$$a(x_{n+1})+b(x_{n+1})+c(x_{n+1})= =a\left(w_n-\frac{f(w_n)}{f'(w_n)}\right)+b\left(w_n-\frac{f(w_n)}{f_n'}\frac{f_n+\beta_1f(w_n)}{f_n+(\beta_1-2)f(w_n)}\right) +c\left(w_n-\frac{f(w_n)}{f_n'}\frac{f_n+\beta_2f(w_n)}{f_n+(\beta_2-2)f(w_n)}\right)$$

Suppose that the coefficients a, b and c is considered as follows:

$$a=1,b=\frac{1}{2(\beta_2-\beta_1)},c=\frac{-1}{2(\beta_2-\beta_1)}$$

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We have:

$$1(w_n)+\frac{1}{2(\beta_2-\beta_1)}(w_n)+\frac{-1}{2(\beta_2-\beta_1)}(w_n)=1\left(x_n-\frac{f_n}{f_n'}\right)+\frac{1}{2(\beta_2-\beta_1)}\left(x_n-\frac{f_n}{f_n'}\right)+\frac{-1}{2(\beta_2-\beta_1)}\left(x_n-\frac{f_n}{f_n'}\right)$$

Because the coefficients b and c are symmetric, in the first side$w_n$coefficient is equal to 1, so we have:

$$w_n=x_n-\frac{f_n}{f_n'}$$

This means that the calculatedw_n at (F) remains unchanged.On the other hand, in second relation will have:

$$1(x_{n+1})+\frac{1}{2(\beta_2-\beta_1)}(x_{n+1})+\frac{-1}{2(\beta_2-\beta_1)}(x_{n+1})=$$

$$=1\left(w_n-\frac{f(w_n)}{f'(w_n)}\right)+\frac{1}{2(\beta_2-\beta_1)}\left(w_n-\frac{f(w_n)}{f_n'}\frac{f_n+\beta_1 f(w_n)}{f_n+(\beta_1-2)f(w_n)}\right)+$$

$$\frac{-1}{2(\beta_2-\beta_1)}\left(w_n-\frac{f(w_n)}{f_n'}\frac{f_n+\beta_2 f(w_n)}{f_n+(\beta_2-2)f(w_n)}\right)$$

Because the coefficients b and c are symmetric,in the first side only x_{n+1} remains and by multiplyingthe coefficients in the parentheses,procedure is as follows:

$$x_{n+1}=w_n-\frac{f(w_n)}{f'(w_n)}-\frac{1}{2(\beta_2-\beta_1)}\frac{f(w_n)}{f_n'}\frac{f_n+\beta_1 f(w_n)}{f_n+(\beta_1-2)f(w_n)}+\frac{1}{2(\beta_2-\beta_1)}\frac{f(w_n)}{f_n'}\frac{f_n+\beta_2 f(w_n)}{f_n+(\beta_2-2)f(w_n)}$$

$$=w_n-\frac{f(w_n)}{f'(w_n)}-\frac{f(w_n)}{f_n'}\frac{1}{2(\beta_2-\beta_1)}\left(\frac{f_n+\beta_1 f(w_n)}{f_n+(\beta_1-2)f(w_n)}-\frac{f_n+\beta_2 f(w_n)}{f_n+(\beta_2-2)f(w_n)}\right)$$

$$=w_n-\frac{f(w_n)}{f'(w_n)}$$

$$-\frac{f(w_n)}{f_n'}\frac{1}{2(\beta_2-\beta_1)}\left(\frac{\big(f_n+\beta_1 f(w_n)\big)\big(f_n+(\beta_2-2)f(w_n)\big)-\big(f_n+\beta_2 f(w_n)\big)\big(f_n+(\beta_1-2)f(w_n)\big)}{(f_n+(\beta_1-2)f(w_n))(f_n+(\beta_2-2)f(w_n))}\right)$$

$$=w_n-\frac{f(w_n)}{f'(w_n)}-\frac{f(w_n)}{f_n'}\frac{1}{2(\beta_2-\beta_1)}\frac{2f^2(w_n)(\beta_2-\beta_1)}{\big(f_n+(\beta_1-2)f(w_n)\big)\big(f_n+(\beta_2-2)f(w_n)\big)}$$

$$=w_n-\frac{f(w_n)}{f'(w_n)}-\frac{f^3(w_n)}{f_n'}\frac{1}{\big(f_n+(\beta_1-2)f(w_n)\big)\big(f_n+(\beta_2-2)f(w_n)\big)}$$

Therefore, a new family can be summarized as the following:

$$(F)\begin{cases} w_n=x_n-\dfrac{f_n}{f_n'}\quad(11-3)\\ x_{n+1}=w_n-\dfrac{f(w_n)}{f'(w_n)}-\dfrac{f^3(w_n)}{f_n'}\dfrac{1}{\big(f_n+(\beta_1-2)f(w_n)\big)\big(f_n+(\beta_2-2)f(w_n)\big)}\end{cases}$$

And the error equation is as follows:

$$\varepsilon_{n+1}=\frac{1}{48}\big(-4F_2^2F_3+\big(6+3(\beta_1+\beta_2)\big)F_2^4\big)\varepsilon_n^5+\cdots. \quad (12-3)$$

The proof is as follows:

By definition, method (F) was defined as a following symbolic form:

$(F)=a(N)+b\big(M_{\beta_1}\big)+c\big(M_{\beta_2}\big).$

Which the coefficients a, b and c is considered as follows:

$$a=1, b=\frac{1}{2(\beta_2-\beta_1)}, c=\frac{-1}{2(\beta_2-\beta_1)}$$

According to relations (3-8), (3-10),errorequation of methods $\big(M_{\beta_1}\big)$, (N) and $\big(M_{\beta_2}\big)$ respectively equal to:

$$\varepsilon_{n+1}=\frac{1}{24}(3F_2^3)\varepsilon_n^4+\frac{1}{12}(2F_2^2F_3-3F_2^4)\varepsilon_n^5+\cdots.$$

$$\varepsilon_{n+1}=\frac{1}{24}(-2F_2F_3+(3+6\beta_1)F_2^3)\varepsilon_n^4+\frac{1}{72}(-4F_3^2+(24+36\beta_1)F_2^2F_3)$$
$$-3F_2F_4-(18+54\beta_1+9{\beta_1}^2)F_2^4)\varepsilon_n^5+\cdots.$$

$$\varepsilon_{n+1}=\frac{1}{24}(-2F_2F_3+(3+6\beta_2)F_2^3)\varepsilon_n^4+\frac{1}{72}(-4F_3^2+(24+36\beta_2)F_2^2F_3)$$
$$-3F_2F_4-(18+54\beta_2+9{\beta_2}^2)F_2^4)\varepsilon_n^5+\cdots.$$

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According Richardson extrapolation, the above equation is multiplied in 1, $\frac{1}{2(\beta_2-\beta_1)}$ and $\frac{-1}{2(\beta_2-\beta_1)}$ respectively. So, we will have:

$$\varepsilon_{n+1} = 1(\frac{1}{24}(3F_2^3) + \frac{1}{2(\beta_2-\beta_1)}(\frac{1}{24}(-2F_2F_3 + (3+6\beta_1)F_2^3) - \frac{1}{2(\beta_2-\beta_1)}\frac{1}{24}(-2F_2F_3 + (3+6\beta_2)F_2^3)\varepsilon_n^4 + K\varepsilon_n^5$$

Where K is a function of $F_2$and F_3. It can be seen easily that the coefficients ofε_n^4will be zero and the proof is complete and this represents the fifth order of convergence of Newton Raphson method.

3-1-2 Choosing appropriate values for the parameters $\beta_1$and β_2

Although it may seem that small value for $\beta_1$and β_2 are suitable, But these values are chosen in such a way that leads to simplicity of iterative process.For example, If$(\beta_1,\beta_2) = (2,4)$is selected by placing these values in equation (3-11) we have:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{\big(f_n + (2-2)f(w_n)\big)\big(f_n + (4-2)f(w_n)\big)} \end{cases}$$

And iterative process is as follows:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{f_n\big(f_n + 2f(w_n)\big)} \end{cases} \qquad (13-3)$$

Using equation (3-12) the error equation is as follows:

$$\varepsilon_{n+1} = \frac{1}{48}(-4F_2^2F_3 + 24F_2^4)\varepsilon_n^5 + \cdots. \qquad (14-3)$$

Now, if $(\beta_1,\beta_2) = (0\ ,2)$ is selected, by substitution of these values into the equation (3-11) will have:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{\big(f_n + (0-2)f(w_n)\big)\big(f_n + (2-2)f(w_n)\big)} \end{cases}$$

And iterative process is as follows:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{f_n\big(f_n - 2f(w_n)\big)} \end{cases} \qquad (15-3)$$

Using equation (3-12) the error equation is as follows:

$$\varepsilon_{n+1} = \frac{1}{48}(-4F_2^2F_3 + 12F_2^4)\varepsilon_n^5 + \cdots.$$

Also, if $\beta_1 = \beta_2 = 2$ is selected, by substitution of these values into the equation (3-11) will have:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{\big(f_n + (2-2)f(w_n)\big)\big(f_n + (2-2)f(w_n)\big)} \end{cases}$$

And consequently we have:

$$\begin{cases} w_n = x_n - \frac{f_n}{f_n'} \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{f^3(w_n)}{f_n'}\frac{1}{(f_n)^2} \end{cases}$$

By multiplication and division of $f_n$will have:

$$\begin{cases} w_n = x_n - \dfrac{f_n}{f_n'} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)} - \dfrac{f_n}{f_n'}\left(\dfrac{f(w_n)}{f_n}\right)^3 \end{cases}$$

Using equation (3-12) the error equation is as follows:

$$\varepsilon_{n+1} = \frac{1}{48}(-4F_2^2F_3 + 18F_2^4)\varepsilon_n^5 + \cdots$$

If the parameters are chosen such that satisfy the condition$\beta_1 + \beta_2 = -2$, the term of F_2^4 in equation (3-12) will be omitted, because the coefficient ofF_2^4 in the error equation is$\left(6 + 3(\beta_1 + \beta_2)\right)$.For example, if $(\beta_1, \beta_2) = (2, -4)$ is selected, which satisfy the condition$\beta_1 + \beta_2 = -2$, using equation (3-11) the iterative process will be as follows:

$$\begin{cases} w_n = x_n - \dfrac{f_n}{f_n'} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)} - \dfrac{f^3(w_n)}{f_n'}\dfrac{1}{\left(f_n + (2-2)f(w_n)\right)\left(f_n + (-4-2)f(w_n)\right)} \end{cases}$$

Which it can be shown in the simple following form:

$$\begin{cases} w_n = x_n - \dfrac{f_n}{f_n'} \\ x_{n+1} = w_n - \dfrac{f(w_n)}{f'(w_n)} - \dfrac{f^3(w_n)}{f_n'}\dfrac{1}{f_n\left(f_n - 6f(w_n)\right)} \end{cases}$$

Using equation (3-12) the error equation is as follows:

$$\varepsilon_{n+1} = \frac{1}{48}(-4F_2^3F_3)\varepsilon_n^5 + \cdots.$$

In the following, numerical examples are presentedto gain root function of $f(x)$ and to ensure the order of convergence of the proposed methods.

Research Findings (numerical example):

Here, computer programs are written using the MATLAB languageand numerical examples forfinding simple zeros of nonlinear functions is presented using different methods including Newton Raphson, modified Newton Raphson, Secant,Jart,fourth-order King,fifth-order King and Dehghan.It should be noted thatall the numerical examples with error 10^{-6} have been performed, and a brief comparison between theirs rate of convergence have been done.

Computer programs:

Newton's method:

A computer program for Newton's methodis as follows:

```
%%%%%%%%%%%Newton%%%%%%%%%%%%%
clc; clear;
start=-7;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start ;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
fper(j)=exp(x(i))-(1/(1+(x(i)^2)));
x(j+1)=x(i)-f(j)/fper(j);
if  abs(abs( x(j+1))- abs(x(i)))<= 0.000001;
break
else
x(i)=x(j+1);
end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'Newt0n');
```

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Modified Newton Raphson method:

A computer program for modified Newton Raphsonis as follows:

```
%%%%%%%%%%%Newton Pirasteh%%%%%%%%%%%%%
clc;clear;
start=-7;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start ;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
fper(j)=exp(x(i))-(1/(1+(x(i)^2)));
fzeg(j)=exp(x(i))+(2*x(i)/(1+(x(i)^2))^2);
x(j+1)=x(i)-((f(j)*fper(j))/((fper(j))^2-f(j)*fzeg(j)));
if  abs(abs( x(j+1))- abs(x(i)))<= 0.000001;
break
else
x(i)=x(j+1);
end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'NewtonPirasteh');
```

Secant method:

A computer program for Secant methodis as follows:

```
%%%%%%%%%%%Ghate%%%%%%%%%%%%%
clc;clear;
start1=-7;
start2=-6.9;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start1 ;
x(i+1)=start2;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
f(j+1)=exp(x(i+1))-1.5-(atan(x(i+1)));
x(j+2)=x(i+1)-(f(j+1)*((x(i+1)-x(i))/(f(j+1)-f(j)))) ;
if  abs(abs( x(j+1))- abs(x(i)))<= 0.000001;
break
else
x(i)=x(j+1);
x(i+1)=x(j+2)
end
end
end
x=[start1 start2  x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'Ghate');
```

Jart method:

A computer program for Jart methodis as follows:

```
%%%%%%%%%%%Jart%%%%%%%%%%%%%
clc;clear;
start=-7;
repeat=100;
```

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```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start ;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
fper(j)=exp(x(i))-(1/(1+(x(i)^2)));
w1=f(j)/fper(j)
fper2(j)=exp(x(i)-2/3*w1)-(1/(1+((x(i)-2/3*w1)^2)));
x(j+1)=x(i)- 0.5*w1+(f(j)/(fper(j)-3*fper2(j)));
if  abs(abs( x(j+1))- abs(x(i)))<= 0.00000001;
break
else
x(i)=x(j+1);

end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'Jart');
```

Fourth-order King:

A computer program for Fourth-order Kingis as follows:

```
%%%%%%%%%%King 4th%%%%%%%%%%%%%
clc;clear;
start=-7;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start ;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
fper(j)=exp(x(i))-(1/(1+(x(i)^2)));
w=x(i)-f(j)/fper(j)
fw=exp(w)-1.5-(atan(w));
x(j+1)=w - (fw/fper(j))*(f(j)/(f(j) -2*fw));
if  abs(abs( x(j+1))- abs(x(i)))<=0.0000000001;
break
else
x(i)=x(j+1);
end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'King 4th');
```

Fifth-order King:

A computer program for Fifth-order Kingis as follows:

```
%%%%%%%%%%%King 5th%%%%%%%%%%%%%%
clc;clear;
start=-7;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start ;
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
```

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```
fper(j)=exp(x(i))-(1/(1+(x(i)^2)));
w=x(i)-f(j)/fper(j)
fw=exp(w)-1.5-(atan(w));
fperw= exp(w)-(1/(1+(w^2)));
u=f(j)-2*fw
fu=exp(u)-1.5-(atan(u));
x(j+1)=w - (fw/fperw) - ( fw^3/fper(j) ) *  (1/fu);
if  abs(abs( x(j+1))- abs(x(i)))<= 0.0000001;
break
else
x(i)=x(j+1);
end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'King 5th');
```

Dehghan's method:

A computer program for Dehghan's methodis as follows:

```
%%%%%%%%%Dehghan%%%%%%%%%%%%
clc;clear;
start=-7;
repeat=100;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
fori=1:1
x(i)=start;
%    z(i)=0
for j=1:repeat
f(j)=exp(x(i))-1.5-(atan(x(i)));
u1=x(i)+f(j);
u2=x(i)-f(j)
fu1=exp(u1)-1.5-(atan(u1));
fu2=exp(u2)-1.5-(atan(u2));
z(j+1)=x(i)+  (2*f(j)^2)/(fu1- fu2);
fz(j)=exp(z(j+1))-1.5-(atan(z(j+1)));
x(j+1)=x(i) - (  2*f(j)*(fz(j)-f(j))   )/  (fu1-fu2);
if  abs(abs( x(j+1))- abs(x(i)))<= 0.0000001;
break
else
x(i)=x(j+1);
z(j)=z(j+1);
end
end
end
x=[start x(2:j)]
f=abs(f)
Result=[x' f']
xlswrite('C:\MATLAB\mat\\P_example1', Result, 'Dehghan');
```

Numerical examples:

Example 4-2-1:

Calculate the root of the function $f(x) = e^x - \frac{15}{10} - tan^{-1}x = 0$ using different iterative methods.

Solution:

Different iterative methods with starting point $x_1 = -7$ and error value 10^{-6} have been considered. The results are shown in the Table 4-1.

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Table 4-1: Results of numerical methods for the equation $f(x) = e^x - \frac{15}{10} - tan^{-1}x = 0$

Numerical methods	*Number of iteration*	x_n	$f(x_n)$
Newton Raphson	*7*	*-14/1 0126977*	*9/01*$\times \mathbf{10^{-12}}$
Secant	*11*	*-14/1 0126977*	*0*
Jart	*4*	*-14/1 0126977*	*3/82962*$\times \mathbf{10^{-10}}$
fourth-order King	*4*	*-14/1 0126977*	*1/37068*$\times \mathbf{10^{-12}}$
fifth-order King	*5*	*-14/1 0126977*	*0*
Dehghan	*5*	*-14/1 0126977*	*2/79776*$\times \mathbf{10^{-14}}$

As it can be seen from Table 4-1, fourth-order King and Jart methods have the higher rate of convergence and in 4 steps the result is close enough to the real roof of the function. The Secant method has the minimum rate of convergence and after 11 steps is close to the real roof of the function.

Example 4-2-2:

Calculate the root of the function $g(x) = cos\ x - xe^x + x^2 = 0$ using different iterative methods.

Solution:

Different iterative methods with starting point $x_1 = -7$ and error value 10^{-6} have been used. The results are shown in the Table 4-2.

Table 4-2: Results of numerical methods for the equation $g(x) = cos\ x - xe^x + x^2 = 0$

Numerical methods	*Number of iteration*	x_n	$g(x_n)$
Newton Raphson	*10*	0/*639154 0 96*	*1/5077*$\times \mathbf{10^{-6}}$
Secant	*14*	0/*639154 096*	*6/26221*$\times \mathbf{10^{-13}}$
Jart	*35*	0/*639154576*	*1/74708*$\times \mathbf{10^{-6}}$
fourth-order King	*7*	0/*639154 096*	*6/60089*$\times \mathbf{10^{-12}}$
fifth-order King	*8*	0/*639154 096*	*3/38356*$\times \mathbf{10^{-10}}$
Dehghan	*10*	0/*639154 0 96*	*6/66134*$\times \mathbf{10^{-10}}$

Because the starting point is not selected sufficiently close to the real root function, theSecant and Jart methods did not work efficiently and a large number of iterations were required. The number of iteration that required for the fourth-order King andfifth-order King Methods was 7 and 8, respectively, and they have an acceptable rate of convergence. This shows that the fourth-order King andfifth-order King Methods have lower sensitivity to the choice of starting point.

Example 4-2-3:

Calculate the root of the function $h(x) = x^5 - 8\,x^4 + 24\,x^3 - 34\,x^2 + 23\,x - 6 = 0$ using different iterative methods.

Solution:

Different iterative methods with starting point $x_1 = 0.5$ and error value 10^{-6} have been used. The results are shown in the Table 4-3.

Table 4-3: Results of numerical methods for the equation$h(x) = x^5 - 8\,x^4 + 24\,x^3 - 34\,x^2 + 23\,x - 6 = 0$

Numerical methods	*Number of iteration*	x_n	$h(x_n)$
Newton Raphson	*29*	0/*999988558*	*0*
Secant	*43*	0/*999993027*	*0*
Jart	*14*	0/*999997816*	*0*
fourth-order King	*19*	0/*999989392*	*0*
fifth-order King	*18*	1/*000010516*	*0*
Dehghan	*14*	0/*999467037*	*3/03022*$\times \mathbf{10^{-10}}$

In this example, the function is a polynomial of degree 5 and Dehghan andJart methods have considerably faster convergence than other methods.The Secant method has the minimum rate of convergence and after 43 iterations is close to the real roof of the function.

Example 4-2-4:

Calculate the root of the function $k(x) = arctg(3x^3 - 1) + 6\,x^3 + e^{5\,x^3} + 80\ = 0$ using different iterative methods.

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Solution:

Different iterative methods with starting point $x_1 = -0.1$ and error value 10^{-6} have been used. The results are shown in the Table 4-4.

Table 4-4: Results of numerical methods for the equation $k(x) = arctg(3x^3 - 1) + 6\,x^3 + e^{5\,x^3} + 80\ = 0$

Numerical methods	*Number of iteration*	x_n	$k(x_n)$
Newton Raphson	*14*	*-1/035401759*	$3/\mathbf{34022} \times \mathbf{10^{-8}}$
Secant	*24*	*-1/035401759*	*9/11271*$\times$ $\mathbf{10^{-3}}$
Jart	*33*	*-1/035401759*	*1/4682*$\times$ $\mathbf{10^{-5}}$
fourth-order King	*8*	*-1/035401759*	*3/55271*$\times$ $\mathbf{10^{-15}}$
fifth-order King	*8*	*-1/035401759*	*3/61615*$\times$ $\mathbf{10^{-8}}$
Dehghan	*6*	*-1/035401759*	*3/87941*$\times$ $\mathbf{10^{-}}$

In this example, Dehghan, fourth-order King and fifth-order King methods reach sufficiently close to the real root function after 6, 8 and 8 iterations, respectively, while the other methods needat least 14 iterations.The Jart method has the minimum rate of convergence.

Result:

Advantage of the Newton Raphson method:

- Have order of convergence 2, and the convergence rate is at a good level.
- Due to the simplicity of the equation, it use in many engineering problems,especially in the distribution of loads on large structures.
- If α is a simple root of *f (x)* and $x_0$is close enough to the root and if Newton's method is convergent, the convergence rate of this method is higher than the Secant method.
- If $f$is an increasing function and the second derivative is always positive, then, the Newton Raphson method, with any starting point, is always convergent.
- If the calculation of the derivative of the function *f*is simple, Newton's method can be used to approximate the root of the function$f(x) = 0$.
- Newton Raphson method can be applied to the k-dimensional as the following:

  $x^{k+1} = x^k - \left(f'(x^k)\right)^{-1} f(x^k)$

  Where$(f'(x^k))$represents the Jacobian and can be used in solving nonlinear systems of equations, while this approach is not feasible for the Secant method.

Disadvantage of the Newton Raphson method:

- Because the calculation of $f'(x)$ is required at each stage, if calculation of the derivative of the function is complex, it is practically impossible to use of this method. Especially in some cases that the derivative of the function nears the root of the function is very close to zero, which may be leads to the divergence of the method.
- In general, there is no guarantee that Newton's method is convergent with any starting point, unless function f is convex.
- In the Newton Raphson method, the data at each step should be stored in computer memory that leads to speed up the computational speed.
- If α is a root of a function *f*with order$m > 1$, then it is possible that the rate of convergence of Newton Raphson methodreduce greatly. Therefore,often in such cases, the modified Newton Raphson method is used.

Advantage of the Secant method:

- Because this method is a special case of Newton Raphson method in which the derivative of the function is replaced with an approximation as follows:

  $\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

  From the engineering and technical point of view, it is very favorable indeed and derivative of the function can be calculated using value of the function *f*at the points $x_n$and x_{n-1}, and thus, it reduces the complexity of the problem.
- The Secant's method in terms of data storage in computer memory is smaller than the Newton'sRaphson method.
- No need to calculate the derivative of the function at each step.
- It has super-linear convergence.

Disadvantage of the Secant method:

- In general, there is no guarantee of convergence of the Secant method.
- Unlike the Newton's method, generally, this method is not used in solving nonlinear systems of equations.
- This method does not the capability to improve the order of convergence.
- It has lower order of convergence than to the Newton Raphson method.

Advantage of the modified king's method:

- Since, as much as possible, the need to directly calculate the derivative of the function at each step can be avoided; it reduces the complexity of the problem and is highly desirable for engineering and technical calculations.
- Since the calculations in each step are also used for the next step, in terms of saving space of memory in computer, the Secant's andNewton's method are efficiently and faster.
- These methods have the order of convergence four and five. Therefore, they have the higher convergence rate than to the Secant's and Newton's methods.
- In each step, the calculation of the derivative of a function and two values of a function is necessary.
- By changing the parameters of this method, a new class that have the order of convergence four and five can be achieved.

Disadvantage of the modified king's method:

- Unlike the Newton's method, generally, this method is not used in solving nonlinear systems of equations.
- If α is a root of a function *f*with order$m > 1$,in this case, these methods cannot be used to approximate the roots of a function.
- Due to the complexity of theerror analysis of these methods, the direct calculation of error equation is not simple, and therefore, a bulky computer programming called Project MAC's Symbolic Manipulation System (MACSYMA) is used,which not only has the abilityto calculate at high volume, but is efficient as the same amount of symbolic computation.

Advantage of the Dehghan's method:

- It has the order of convergence four, and it has a much more convergence rate than to the Newton's method.
- Since it is a two-step method,first, the initial value is improved using the modified Newton's methodand then in the second stage its convergence improved.
- In this method, since the calculation of the derivative of the function has been deleted, it reduces the complexity of the problem.

Disadvantage of the Dehghan's method:

- This method cannot be used in solving nonlinear systems of equations.
- If the starting point is not sufficiently close to the root,it may be leads to the divergence of the method.
- This method does not the capability to improve the order of convergence.

Advantage of the NTA method:

- Since this method does not need to calculate the second derivative of the function at each step, if the calculation of the second derivative of the function is complex, the use of this method is much more common.
- If$\alpha$is a repeated root of rank *m*of the function *f*,this method is more efficient than other methods.
- In each step, the calculation of the derivative of a function and two values of a function is necessary.
- It is not depend on the calculation of the rank of repeated root of the function, so in general, itcan be very useful where that obtaining of the rank of repeated root is difficult.
- This method has the order of convergence three, therefore, it will be more efficient than modified Newton's method.

Disadvantage of the NTA method:

- It depends on the starting point, which must be calculated using Newton's method.

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