Solving bi-level linear fractional programming problem by bi-level linear programming problem

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ABSTRACT

In this paper, we are going to solve bi-level linear fractional programming problem (BLFPP). Many approaches have been offered to solve the mentioned problem; however most of the suggested methods are just some techniques to solve the problem and they do not have a reliable theoretical background. In this paper, through expanding the variable transformation of Charnes and Cooper (1962), BLFPP will change to bi-level linear programming problem (BLPP). Considering that there are some efficient and theory-based algorithms to solve the BLPP, BLFPP is solvable, as well. In this paper, the k-th best method, which is one of the most applicable and popular methods to solve BLPP, is used as a technique to solve the obtained BLPP. A numerical example will be given, at the end, to explain the method.

INTRODUCTION

Bi-level programming problem is a decentralized problem. In this problem, two decision makers are existed in the hierarchical construction, too. Each decision maker controls individually decision variables. The upper level in the hierarchy can just influence the lower level. This problem has a lot of applications in management, economics, etc. As an applicable example, one can consider an exporter country which concentrates on two important products 1 and 2. Two decision making levels are related to this situation, the government (the upper level), and the manager of the institute (the lower level); each of which controls only one individual decision variable. Therefore, the government controls the variable which can optimize the expert level and the institute can choose the other variable to optimize its own profits. In this case, two objectives will be created which belong to expert optimization and profit optimization. In the last two decades much research has been carried out at solving bi-level programming problem (Bialas and Karwan, 1984; Bard and Moore, 1990; Anandalingam and White, 1990; Liu and Spencer, 1995).

Bi-level linear programming problem (BLPP) is a bi-level program, in which all the objective functions and constraints are linear (Bialas and Karwan, 1984; Liu and Spencer, 1995; Candler and Karwan, 1982). Bi-level linear fractional programming problem (BLFPP) has a similar construction to BLPP; the only difference is that the objective functions are fractional, in this case. There are some limited methods to solve BLFPP. BLFPP concepts were for the first time offered by Calvate and Cale (1999). Then in 2004, they (Calvate and Cale, 2004) modified the paper which was previously stated by Arora and Thirwan (1993). Weighting method which was offered by Mishara (2007) is to find the appropriate weights and through the hierarchical process, model is changed the hierarchy to optimization, in a way that the objective functions in two levels convert to one objective function. In 2009 a paper was carried out about 0-1 BLFPP with independent followers by Narang and Arove (2009). Their paper was based on a method through which 0-1 BLFPP with independent followers converts to 0-1 BLFPP with just one independent follower. Then applying a variable transformation on the upper and lower level problems, it will change to a BLPP. Finally, the solutions will be obtained by k-th best method. Taylor series method was offered by Duran Toksari (2010). In this method, first the optimal solution of each fractional objective function is achieved. Then, for each fractional function the expanding of Taylor series on the optimized solution is written. After that, determining the appropriate weight to each function, BLFPP changes to a BLPP. In 2011 a paper was published named "a method based on fuzzy goal programming" by Pramanik and Pratim-Day (2011); in which determining the best solution for objective function, the fractional membership function is made. Then, by approximation of Taylor series, the fractional membership functions

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change to the equivalent linear membership functions. At last, determining the best solution for decision variables, fuzzy programming problem is formulated.

Most of the methods mentioned to solve BLFPP are just some techniques to solve BLFPP. In this method, BLFPP is converted to a problem with another nature (BLFPP is transformed to a linear programming problem (LPP)), which is not realistic. Therefore, in this paper expanding Charnes and Cooper’s variable transformation (Charnes and Cooper, 1962), BLFPP converts to BLPP, which does not include the previous methods, for, it is shown that the Bi-level nature of the problem is served. In this situation, all the offered methods to solve BLPP are applicable on it, and the optimal solution of BLFPP is satisfactorily achieved.

This paper includes the following sections; bi-level programming is introduced in section 2. In section 3 the variable transformation method is expanded to solve BLFPP. Section 4 belongs to a numerical example to clarify the mentioned method. The last section includes the results and conclusions.

Bi-level programming problem:
An introduction to Bi-level programming:

Bi-level programming is a decentralized programming, in which there are two decision makers in its hierarchical construction. In this model, the upper level has influence the lower level.

Bi-level programming problem can be briefly expressed as follows:

\[
\begin{align*}
\max_{x_1} & \quad \mathbf{Z}_1(x_1, x_2), \\
x_1 & \in X_1 = \{x_1 : x_1 \geq 0\}, \\
& \text{in which } x_2 \text{ is optimal solution of the following problem:} \\
\max_{x_2} & \quad \mathbf{Z}_2(x_1, x_2), \\
x_2 & \in X_2 = \{x_2 : x_2 \geq 0\}, \\
& \text{s.t. } g(x_1, x_2) \leq 0, \\
& \text{in which } Z_1, Z_2 \text{ and } g \text{ are functions as } Z_1 : X_1 \times X_2 \to \mathbb{R}, \ Z_2 : X_1 \times X_2 \to \mathbb{R} \text{ and } g : X_1 \times X_2 \to \mathbb{R}^m.
\end{align*}
\]

In this problem, the upper level controls \(x_1\) and the lower level controls \(x_2\). First, decision maker in the upper level chooses an \(x_1\). Then, the lower level reacts and chooses an \(x_2\) in which \((x_1, x_2)\) is a feasible solution for the problem.

Definition 1. Suppose \(S\) is a set of such \((x_1, x_2)\).

Definition 2. \(S^\prime\) includes \((x_1, x_2)\) in which \(x_1\) is an arbitrary solution in \(S\) and \(x_2\) is the optimized point of the lower level corresponding each optional \(x_1\).

Definition 3. Point \((x_1^*, x_2^*)\) is the optimized solution of BLPP if:

1. \((x_1^*, x_2^*) \in S^\prime\).
2. For each feasible solution \((\overline{x}_1, \overline{x}_2) \in S^\prime : Z_1(x_1^*, x_2^*) \geq Z_1(\overline{x}_1, \overline{x}_2)\).

An introduction to BLFPP:

BLFPP is a bi-level problem in which each objective function is a linear fractional function. For example, suppose the upper level objective function is export/import ratio in a country, and also, the lower level objective function is profit/cost ratio in a company.

A BLFPP is formulated as follows:

\[
\begin{align*}
\max_{x_1} & \quad \mathbf{Z}_1(x_1, x_2) = \frac{c_{11}x_1 + c_{12}x_2 + c_{13}}{d_{11}x_1 + d_{12}x_2 + d_{13}}, \\
x_1 & \in X_1 = \{x_1 : x_1 \geq 0\}, \\
& \text{in which } x_2 \text{ is optimal solution of the following problem:}
\end{align*}
\]
\[
\begin{align*}
\max Z_2(x_1, x_2) &= \frac{c_{21}x_1 + c_{22}x_2 + c_{23}}{d_{21}x_1 + d_{22}x_2 + d_{23}}, \\
\text{s.t.} \quad g(x_1, x_2) &= A_1x_1 + A_2x_2 - \overline{b} \leq 0, \quad (2)
\end{align*}
\]

in which \( Z_1, Z_2 \) and \( g \) are functions as \( Z_i : X_1 \times X_2 \rightarrow \mathbb{R} \). Also, \( X_1 \subseteq \mathbb{R}^{n_1}, X_2 \subseteq \mathbb{R}^{n_2} \) and \( c_{11}, c_{21}, d_{11}, d_{21} \in \mathbb{R}^{n_1}, c_{12}, c_{22}, d_{12}, d_{22} \in \mathbb{R}^{n_2} \) and \( \overline{b} \in \mathbb{R}^{m} \), and \( c_{13}, c_{23}, d_{13}, d_{23} \) are scalars. \( A_1 \) and \( A_2 \) are matrices with dimensions \( m \times n_1 \) and \( m \times n_2 \), respectively. It is also supposed that the denominators of the fractions are positive for each pair \((x_1, x_2)\) satisfied on the constraints of model (2). If the denominators of the fractions are considered equal to one, the problem is transformed to a BLPP.

**Expanding Charnes and Cooper’s variable transformation:**

In this section, a variable transformation is used to convert BLFPP into BLPP. Considering model (2), the following variable is defined:

\[
t = \min \left\{ \frac{1}{d_{11}x_1 + d_{12}x_2 + d_{13}}, \frac{1}{d_{21}x_1 + d_{22}x_2 + d_{23}} \right\}.
\]

So,

\[
t \leq \frac{1}{d_{11}x_1 + d_{12}x_2 + d_{13}}, \tag{3}
\]

and

\[
t \leq \frac{1}{d_{21}x_1 + d_{22}x_2 + d_{23}}. \tag{4}
\]

By variable transformation \( y_1 = tx_1 \) and \( y_2 = tx_2 \), relations (3) and (4) convert to:

\[
d_{11}y_1 + d_{12}y_2 + d_{13}t \leq 1,
\]

and

\[
d_{21}y_1 + d_{22}y_2 + d_{23}t \leq 1.
\]

Applying mentioned variable transformation on the constraints and objective functions of model (2), the following BLPP is obtained:

\[
\begin{align*}
\max Z_1(y_1, y_2) &= c_{11}y_1 + c_{12}y_2 + c_{13}t, \\
y_1 \in Y_1 &= \{y_1 : d_{11}y_1 + d_{12}y_2 + d_{13}t \leq 1, y_1 \geq 0\},
\end{align*}
\]

in which \( y_2 \) and \( t \) are the optimal solution of the following problem:

\[
\begin{align*}
\max Z_2(y_1, y_2) &= c_{21}y_1 + c_{22}y_2 + c_{23}t, \\
(y_2, t) \in Y_2 &= \{(y_2, t) : d_{21}y_1 + d_{22}y_2 + d_{23}t \leq 1, y_2 \geq 0, t \geq 0\},
\end{align*}
\]

s.t. \( g(y_1, y_2) = A_1y_1 + A_2y_2 - \overline{b}t \leq 0 \),

This is a BLPP equivalent to the BLFPP (2). Thus, by solving the problem (5), the optimal solution of the primal model is acquired.

**Example:**

Consider the following BLFPP as:

\[
\begin{align*}
\max Z_1(x_1, x_2) &= \frac{x_1 + 2x_2 + 2}{x_1 + x_2 + 1}, \\
x_1 \in X_1 &= \{x_1 : x_1 \geq 0\},
\end{align*}
\]
\[
\max_{x_2} Z_2(x_1, x_2) = \frac{2x_1 + x_2 - 1}{x_1 + 2x_2 + 3},
\]
\[x_2 \in X_2 = \{x_2 : x_2 \geq 0\},\]
s.t. \(-x_1 + 2x_2 \leq 3,\)
\[2x_1 - 3x_2 \leq 3,\]
\[x_1 + x_2 \geq 3\]
(6)

The feasible region of problem (6) can be represented in the following figure:

**Fig. 1:** Feasible region and objective functions of model (6)

One can distinguish \(\overline{S}\) based on the Figure 1 in which it is indicated with bold lines. The optimal solution of the upper level on \(\overline{S}\) is the extreme point \(B\). Now, we solve BLFPP (6) by the proposed method in this paper. Based on the mentioned method in section 3, the following variable transformation is applied on the problem, we have:

\[
t = \min \left\{ \frac{1}{x_1 + x_2 + 1}, \frac{1}{x_1 + 2x_2 + 3} \right\}
\]

So,
\[
t \leq \frac{1}{x_1 + x_2 + 1}
\]
and
\[
t \leq \frac{1}{x_1 + 2x_2 + 3}
\]

The transforming the variables \(y_1 = tx_1\) and \(y_2 = tx_2\), we have:
\[
y_1 + y_2 + t \leq 1
\]
and
\[
y_1 + 2y_2 + 3t \leq 1
\]

Applying the mentioned variable transformation on the objective functions and constraints, the following BLPP is obtained:

\[
\max_{y_1} Z_1(y_1, y_2) = y_1 + 2y_2 + 2t,
\]
\[y_1 \in Y_1 = \{y_1 : y_1 \geq 0, y_1 + y_2 + t \leq 1\},\]
\[
\max_{y_2,t} Z_2(y_1, y_2) = 2y_1 + y_2 - t,
\]
\[
(y_2, t) \in \mathcal{Y}_2 = \{y_2 : y_2 \geq 0, t \geq 0, y_1 + 2y_2 + 3t \leq 1\},
\]
\[
s.t. -y_1 + 2y_2 - 3t \leq 0,
2y_1 - 3y_2 - 3t \leq 0,
-y_1 - y_2 + 3t \leq 0
\]
Now, the k-th best algorithm is used to solve this problem. The steps are explained as:

**Step 1.** Suppose \( i = 1 \) and the following LPP is solved through simplex method:

\[
\begin{align*}
\max & \quad 2y_1 + 2y_2 + 2t, \\
s.t. & \quad y_1 + y_2 + t \leq 1, \\
& \quad y_1 + 2y_2 + 3t \leq 1, \\
& \quad -y_1 + 2y_2 - 3t \leq 0, \\
& \quad 2y_1 - 3y_2 - 3t \leq 0, \\
& \quad -y_1 - y_2 + 3t \leq 0, \\
& \quad y_1, y_2, t \geq 0
\end{align*}
\]

The optimal solution of this problem is \( Z_{[1]} = \left( \frac{5}{12}, \frac{1}{4}, \frac{1}{36} \right) \). This is the best solution.

Consider \( T = \emptyset \) and \( W = \{ Z_{[1]} \} \).

**Step 2.** Here the algorithm determines if \( Z_{[1]} \) is a member of \( \bar{S} \) or not. The answer to this question is achieved by solving the below problem, through the simplex method.

\[
\begin{align*}
\max & \quad 2y_1 + y_2 - t, \\
s.t. & \quad y_1 + y_2 + t \leq 1, \\
& \quad y_1 + 2y_2 + 3t \leq 1, \\
& \quad -y_1 + 2y_2 - 3t \leq 0, \\
& \quad 2y_1 - 3y_2 - 3t \leq 0, \\
& \quad -y_1 - y_2 + 3t \leq 0, \\
& \quad y_1, y_2, t \geq 0
\end{align*}
\]

The optimal solution to the above model is \( \bar{Z} = \left( \frac{5}{12}, \frac{5}{18}, 0 \right) \) in which \( Z_{[1]} \neq \bar{Z} \). So, \( Z_{[1]} \not\in \bar{S} \).

**Step 3.** The extreme points besides \( Z_{[1]} \) include \( \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{8} \right), \left( \frac{4}{11}, \frac{1}{11}, \frac{5}{11} \right) \) and \( \left( \frac{5}{6}, 0, 0 \right) \). So,

\[
W_{[1]} = \left\{ \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{8} \right), \left( \frac{4}{11}, \frac{1}{11}, \frac{5}{11} \right), \left( \frac{5}{6}, 0, 0 \right) \right\}
\]

\[
T = T \cup \{ Z_{[1]} \} = \left\{ \left( \frac{5}{12}, \frac{1}{4}, \frac{1}{36} \right) \right\}
\]

**Step 4.** Put \( i = i + 1 = 2 \). Then, \( Z_{[2]} \) is chosen as optimal solution; the maximum solution of the first objective on \( W_{[1]} \). It equals \( \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{8} \right) \) which is the second best solution.
Step 2. The following LPP is solved:
\[
\begin{align*}
\text{max} & \quad 2y_1 + y_2 - t, \\
\text{s.t.} & \quad y_1 + y_2 + t \leq 1, \\
& \quad y_1 + 2y_2 + 3t \leq 1, \\
& \quad -y_1 - 2y_2 - 3t \leq 0, \\
& \quad 2y_1 - 3y_2 - 3t \leq 0, \\
& \quad -y_1 - y_2 + 3t \leq 0, \\
& \quad y_1, y_2, t \geq 0, \\
& \quad y_1 = \frac{1}{8}
\end{align*}
\]

(10)
The optimal solution of this problem includes \( Z = \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{8} \right) \) that \( Z_{(2)} = \left( \frac{1}{8}, \frac{1}{4}, \frac{1}{8} \right) \). So, \( Z_{(2)} \) is the optimal solution of this problem, and \( k^* = 2 \).

Conclusion:
In this paper, based on a linearization process, BLFPP was converted to an equivalent BLPP. In this situation, the obtained model (BLPP) has similar nature to the primal model. Because, there are a variety of methods to solve BLLP, we not need to search a method to solve BLFPP. Through the mentioned method in this paper, BLFPP with integer variable and also bi-level linear fractional multi-objective programming problem can be solved. The researchers of this paper will use this method lately to solve BLFPP with fuzzy and interval data.

REFERENCES