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A New Model for Dynamics of Sine-Gordon solitons in The Presence of Perturbation

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A B S T R A C T

An analytical model for adding a space dependent potential to Sine-Gordon field is presented by constructing a collective coordinate for solitary solution of this model. The dynamical behavior of Sine-Gordon soliton in the presence of weak delta function potential barrier and also delta function potential well has been examined analytically. Most of the characters of interaction are derived analytically while they are calculated by direct solution of Sine-Gordon equation numerically. We find that the behavior of a solitary solution is like a point particle which is moved under the influence of a complicated effective potential.

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INTRODUCTION

It is well recognized that the study of nonlinear equations and their solutions is of great importance in many areas of physics. The importance of non-linear wave equations which admit large-amplitude solitary-wave or soliton solutions is that they retain their shape during propagation [HYPERLINK \l "JHA07" 1, HYPERLINK \l "KJa08" 2]. Such solutions have recently received considerable attention by elementary-particle physicists since they may be regarded as extended particle-like solutions of nonlinear field equations.

Dynamical evolution of a field in a presence of impurities, in which case the parameters of the model are functions of space is an important phenomenon from the mathematical point of view and also because of its applications. Impurities and their effect on the motion of solitary-waves must be considered when the dynamics of such solutions are important in the model 3]**}**. In addition, the effect of external field in the motion of solitary-waves is important in theories of the dynamical properties of certain systems characterized by nonlinear wave equations. An external potential can be added to equation of motion as perturbation terms. These effects can also be taken into account by making some parameters of equation of motion to be function of space and time [HYPERLINK \l "KJa08" 2, HYPERLINK \l "Kur11" 4, HYPERLINK \l "KJa01" 5].

It is well-known that when waves scatter on a potential, they can be partly reflected and partly transmitted. For fields with solitonic solutions, the situation is more complicated as solitons cannot split and thus must either bounce, pass through or become trapped inside the potential. This behavior is very sensitive to the value of all the parameters of the model as well as to the initial conditions for the scattering. Most of the researches are based on numerical studies because of non-integrable nature of these systems. So suitable models with analytic solutions are clearly needed to test the validity of such phenomenon and predict their behavior 2,**Error! Bookmark not defined.**

In this paper the model of collective coordinate for investigating the effect of perturbation on the motion of solitary-wave solutions of nonlinear wave equations was described. The method was illustrated by examining the motion of soliton solutions of the Sine-Gordon equation in presence of perturbation and will show its agreement with numerical studies. The particular choice of the Sine-Gordon equation is not crucial. This method can even be used for other field theories.

The paper organized as follows. Sec II is a review of necessary properties of the Sine-Gordon equation and its solutions. And then perturbation described by considering the effect of a weak impurity potential on the motion of Soliton solutions of the Sine-Gordon equation in Sec III. Finally the summary of the results was described and concluded that the Soliton behave as a classical particle which obeys Newton’s second laws of motion.

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II) Basic concepts of Sine-Gordon equation

In this section we first review the properties of the Sine-Gordon equation and some of its solutions. The Sine-Gordon equation is of the form [HYPERLINK \l "JHA07" 1, HYPERLINK \l "Fog77" 3]

$$\frac{\partial^2\psi}{\partial t^2} - c_o^2 \frac{\partial^2\psi}{\partial x^2} + \omega_o^2 \sin(\psi) = 0 \tag{2.1}$$

Where $\psi = \psi(x,t)$ }, c_o is a characteristic velocity and ω_o is a characteristic frequency.

One class of large amplitude traveling-wave solutions of Eq. (2.1) are

$$\psi_{\pm}^{v} = 4\tan^{-1}\left(\exp\left(\pm\frac{\omega_o}{c_o}\gamma(x - vt)\right)\right) \tag{2.2}$$

Where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c_o^2}}}$. These solutions are referred to as solitons (+ sign) and anti-solitons (-sign).

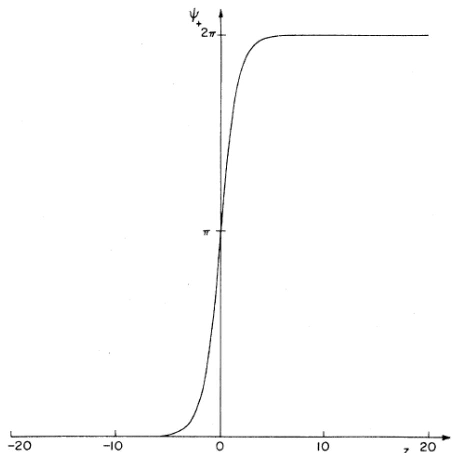

Fig. 1: single Soliton (2.2)

Note that v must be less than c_o in magnitude. These solutions retain their shape during propagation and hence may be classified as solitary wave. The velocity v is a free parameter subject only to the restriction $|v| < c_o$ and thus the soliton may be regarded as a relativistic free particle in the absence of perturbation.

Scattering of solitons from potential have been studied in many papers. The potential generally comes from medium properties. The effects of medium disorders or impurities can be added to equation of motion as perturbative terms, or can be taken into account by making some parameters of Lagrangian as a function of space or time. In this paper we use the last one.

III) Collective coordinate system for dynamics of soliton

Now we examine the effect of an impurity potential on the motion of a soliton initially moving with velocity v . Consider a scalar field with the Lagrangian

$$L = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - U(\psi) \tag{3.1}$$

We consider the Hamiltonian and the Lagrangian in one dimension as the following

$$H = \int dx \left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 + (1-\cos\psi) - \lambda\left(\frac{\partial\psi}{\partial x}\right) g(x)\right] \tag{3.2}$$

$$L = \int dx \left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 - (1-\cos\psi) + \lambda\left(\frac{\partial\psi}{\partial x}\right) g(x)\right] \tag{3.3}$$

where $c_o = \omega_o = 1$, and λ is the coupling constant. The coupling constant is small and may be either positive or negative [4]. The first three terms in Eq.(3.3) comprise the usual Sine-Gordon Hamiltonian (or Lagrangian) density. The last term represent the field-impurity interaction, which we take $g(x)$ in a simple form [3]

$$g(x) = \Theta(x - x_o) - \Theta(x + x_o) \tag{3.4}$$

where Θ is Heaviside step function.

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{3.5}$$

in collective coordinate system the center of soliton is considered as a particle. Thus the collective coordinate could be related to the potential by using one of the models in Sec. II. In this paper we work on Sine-Gordon model with soliton solution of the form Eq.(2.2)

$$\psi = 4\tan^{-1}\left(\exp\left(x - X(t)\right)\right) \tag{3.6}$$

where $X(t) = vt + x_o$ denotes collective coordinate. By inserting Eq.(3.6) into the Lagrangian density (3.3) with using adiabatic approximation we have

$$L = 4\dot{X}^2 - 8 - 4\lambda\left[\tan^{-1}\left(\exp\left(x_o - X\right)\right) - \tan^{-1}\left(\exp\left(-x_o - X\right)\right)\right] \tag{3.7}$$

we use Eq.(3.8) to obtain Eq.(3.7)

$$\int \frac{df(x)}{dx}\Theta(x) = \begin{cases} 0 & x < 0 \\ f(x) - f(0) & x \geq 0 \end{cases} \tag{3.8}$$

Note that $X(t)$ remains as a collective coordinate when integration in Eq.(3.3) performed [4,7,6]. The equation of motion for variable X results from Eq.(3.7)

$$8\ddot{X} + 4\lambda\frac{\sinh(x_o)\sinh(X)}{\cosh^2(X) + \sinh^2(x_o)} = 0 \tag{3.9}$$

This equation shows the effect of non-zero but small values of coupling constant on the soliton solution. For large velocities, the perturbation on the soliton is small as expected, while it is significant modification of the soliton solution for low velocities. The above equation of motion is the same as Fogel et. all [HYPERLINK \l "Fog77" 3].

By employing Eq. (3.9), we obtain the potential energy of the soliton analytically as a function of collective coordinate X }

$$V(X) = 8 + 4\lambda\tan^{-1}\left(\frac{\sinh(x_o)}{\cosh(X)}\right) \tag{3.10}$$

The first term in Eq.(3.10) is just the rest energy of the soliton. And the second term is the change in the static soliton energy due to the impurity. Change in energy can be defined by

$$\Delta V = 4\lambda\tan^{-1}\left(\frac{\sinh(x_o)}{\cosh(X)}\right) \tag{3.11}$$

Eq. (3.11) has an extremum at $X = 0$ (at the center of impurity potential) as sown in figure 2. For $\lambda < 0$, the extremum is a minimum while it is a maximum for $\lambda > 0$. We see that from energy point of view that for negative coupling constant, the impurity attracts soliton and repels anti-soliton and vice versa for positive one.

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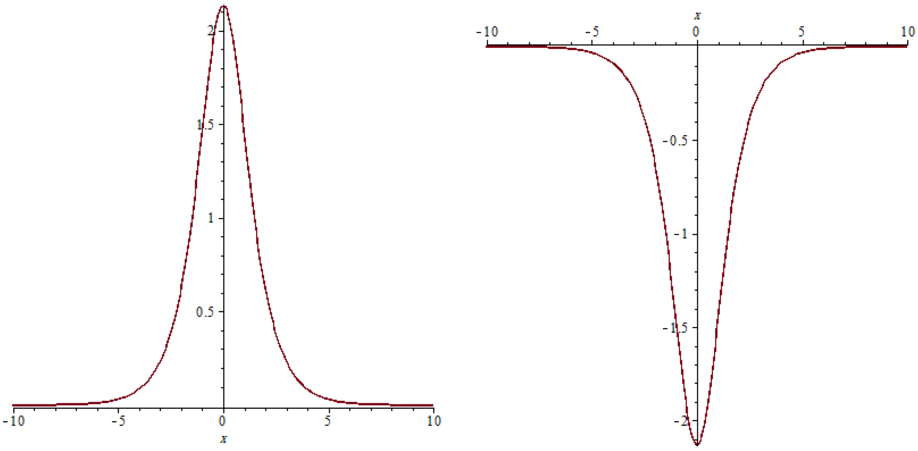

Fig. 2: For $\lambda < 0$, the extremum is a minimum while it is a maximum for $\lambda > 0$.

When the soliton is far from the impurity ($X \to \infty$), the energy of the soliton reduces to $E = 4\dot{X}^2 + 8$, which is the energy of a particle of mass 8 [4,7]. Some other features of the soliton behavior can be found analytically from equations (3.5)-(3.11).

IV) Oscillation frequency in potential well:

Negative values of the coupling constant, i.e. soliton is trapped or bound to the impurity potential, was investigated in this section. One can expect the soliton to execute ancillary motion about $X = 0$. This kind of motion can be easily studied in detail in two different regimes [3]

When the spatial extent of the soliton is small compared to the impurity.

When soliton is extended over a large region compared to the impurity.

In the first case, the soliton should behave as a point particle and for small energies execute harmonic motion. Oscillation frequency of the soliton can be obtained directly from the equation of motion equation (3.7)

$$\int_0^T dt = \int_{\text{near}X=0} \frac{dX}{\sqrt{\lambda \tan^{-1}\left(\frac{\sinh(x_o)}{\cosh(X)}\right) + v_o^2}} \tag{3.12}$$

Where v_o is the initial velocity of the soliton. The result of Eq. (3.10) for $\lambda = -0.2, v_o = 0.00002$ and $x_o = 1.0$ is 0.22207 for oscillation frequency of the soliton, which agree with Fogel result 0.22216 [3]. This means that we are able to see the soliton behavior in impurity potential analytically. Figure 3 shows the oscillation of the Soliton in potential well with $x_0 = 0.005$.

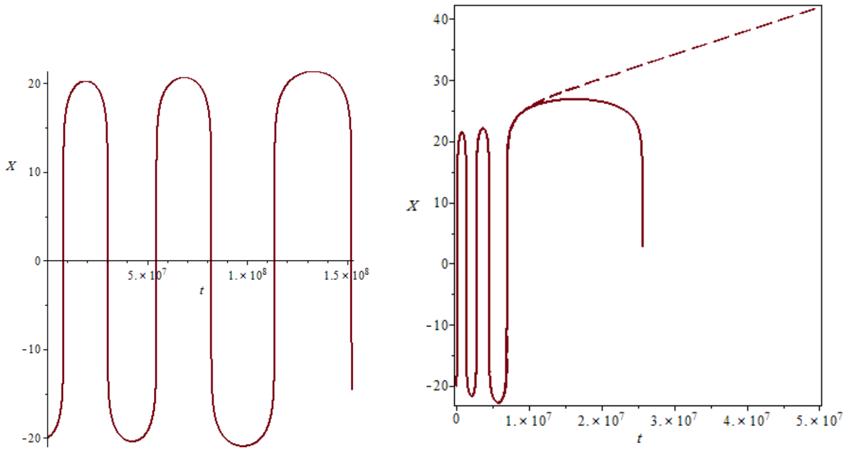

Fig. 3: a. oscillation of the Soliton under the effect of impurity with x0=0.005 b. Soliton escape from potential well.

V) Soliton-barrier interaction:

A soliton-barrier system is modeled with collective coordinate [4]. Consider a soliton with initial velocity of $\dot{X}$ at initial position $X_o = -\infty$. Equation of motion (3.9) shows that the soliton reaches infinity again with the velocity $\dot{X} = \pm\dot{X}_o$. Soliton goes to $-\infty(+\infty)$ if its velocity is less (more) than critical velocity v_c.

$$v_c = \sqrt{\lambda\left[\arctan\left(\frac{\sinh(x_o)}{\cosh(X_o)}\right) - \arctan\left(\frac{\sinh(x_o)}{\cosh(X)}\right)\right]} \qquad (3.13)$$

where x_o is the barrier width.

If $\dot{X}_o < v_c$ then there is a return point in which the velocity of Soliton is zero.

$$\cosh(X_{stop}) = \sinh(x_o)\tan\left(\arctan\left(\frac{\sinh x_o}{\cosh X_o}\right) - \frac{X_o^2}{\lambda}\right) \qquad (3.14)$$

Fig.4 :shows the Soliton-barrier interaction with critical condition.

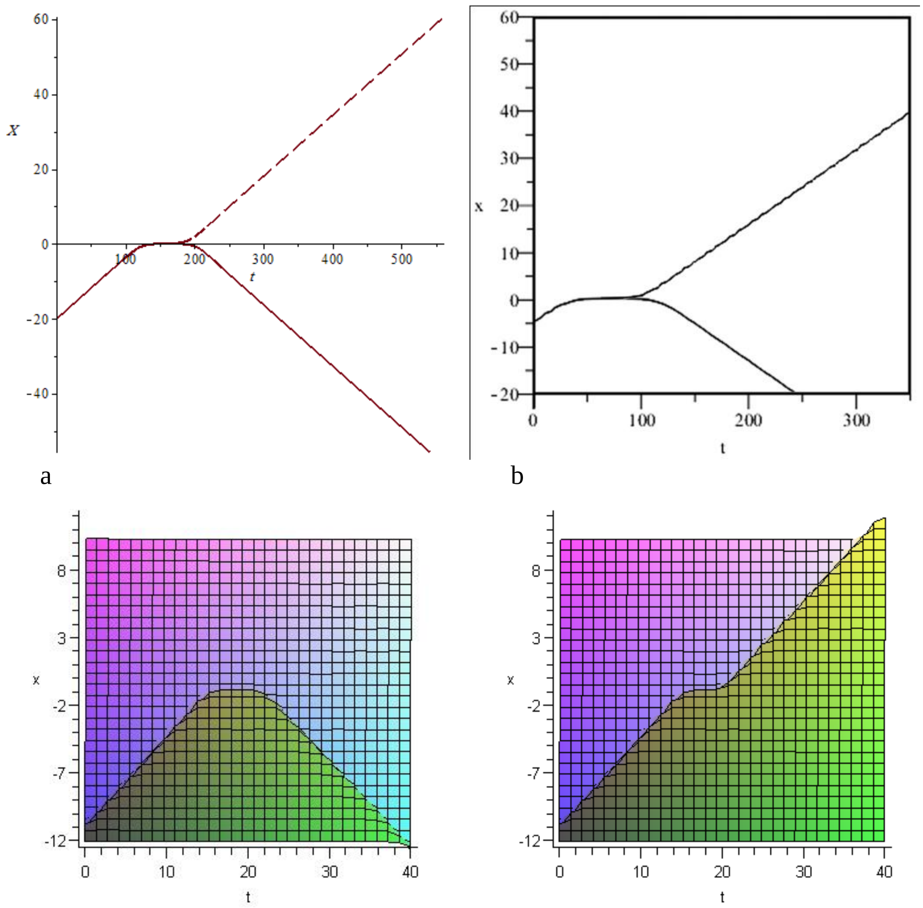

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Fig. 4: Soliton trajectory during the interaction with potential barrier with v:=0.6591 and v:=0.6592. a)collective coordinate model. b) direct solution.

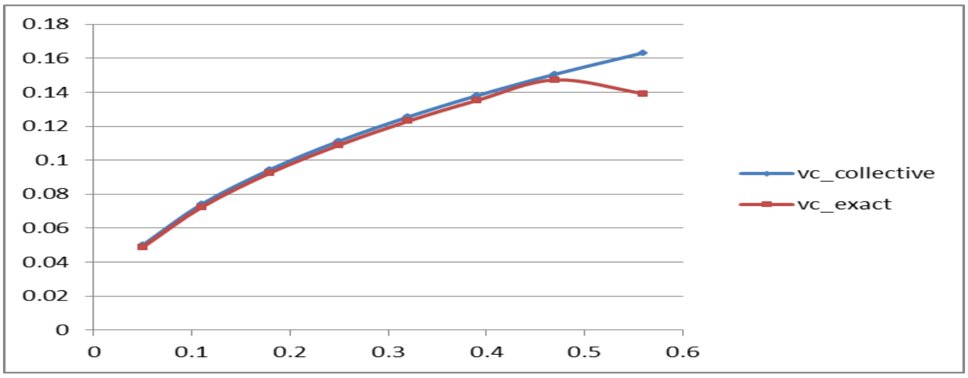

Fig. 5: Critical velocity as a function of x0 with results of direct simulation and analytic model.

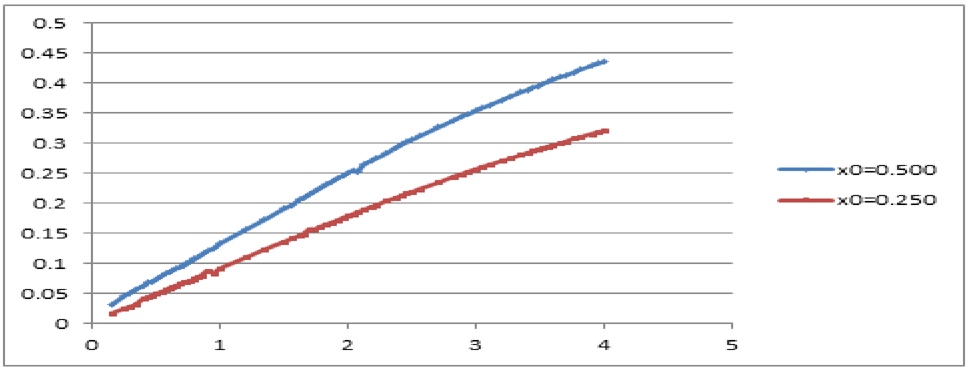

Fig. 6: Critical velocity as a function of barrier height with results of analytic model.

VI) Conclusion and Remarks:

A model for the Sine-Gordon field potential interaction has been presented. Several features of soliton-potential characters were calculated using this model. Calculated characters have been compared with the results of direct simulation of soliton-potential system. In this paper, we have found a critical velocity for the soliton during the interaction with a potential barrier as a function of its initial conditions and the potential characters. The model predicts specific relations between some functions of initial conditions and other functions of final state of the field after the interaction. An escape velocity has been derived for the soliton-well system.

All of the simulations for the soliton-barrier and soliton-well systems show the validity of analytic results. In a previous study, we have applied the same analytic model for sine-Gordon solitons. Therefore, we can conclude that the presented collective coordinate method can be used for most of the solitonic systems. Although this model is able to explain most of the features of the system, but it is unable to explain fine structure of the islands of trapping in soliton-well system. This phenomenon is a very interesting feature of soliton-potential systems. Perhaps one can find an acceptable explanation for this behavior using a further improved version of the collective coordinate method.

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