Some Fixed Point and Common Fixed Point Theorems in G-Cone Metric Spaces

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ABSTRACT

Background: In this paper the concept of G-cone metric is introduced. Objective: Also some results of common fixed point is expressed for mappings which satisfies in weakly compatible condition. Results: Are presented the results were extension of some existing results.

INTRODUCTION

Fixed point theory plays a major role in mathematics and applied sciences, such that optimization, economy and medicine. In the past two decades, fixed point theory concepts quickly expanded. For example in 2007 Huang and Zhang in [4] introduced the concept of cone metric spaces and proved some fixed point theorems for contractive mappings. Then in 2010 Beg and Abbas and Nazir in [1] presented G-cone metric spaces. More results are obtained for fixed point in this spaces. (see for example [1, 5, 6, 7, 8, 9, 11]). The purpose of this paper is applying suitable conditions for two mappings to obtained common fixed point on G-cone metric spaces. To obtain these results some preliminary results and definitions were referred. However for more details see also [2, 4, 6, 9, 12, 13].

2. Preliminaries:

Definition 2.1 ([4]). Let E be a real Banach space and P a subset of E. P is called a cone if and only if:
(i) P is nonempty, closed, and $P 
eq \{0\}$.
(ii) If $a, b \in E$, $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$.
(iii) If both $x \in P$ and $-x \in P$ then $x = 0$.

Given a cone $P \subseteq E$, it will be defined a partial ordering which respect to P by $x \preceq y$ if $y - x \in P$. It will be written $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{Int} P$, where Int P denotes the interior of P. The cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$, $0 \leq x \preceq y \Rightarrow \|x\| \leq K\|y\|$.

The least positive number $K$ satisfying the above statement is called the normal constant of P. The cone P is regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq \ldots \preceq y$, for some $y \in E$, then there is $x \in E$ such that $\|x - x_n\| \to 0$ as $n \to \infty$.

Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 2.1 ([8]). Every regular cone is a normal cone.

Lemma 2.2. Let E be a real Banach space with a cone P. Then:
(i) If $x \preceq y$ and $0 \leq a \leq b$, then $ax \preceq by$.
(ii) If $x \preceq y$ and $u \preceq v$, then $x + u \preceq y + v$.

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Lemma 2.3. Let \( a,b \in \mathbb{R}, a,b \geq 0 \) and, \( x,y \in P \). If \( ax + by \leq 0 \), then \( x = y = 0 \).

Proof. According to property of \( P \), \( ax, by \in P \) and also from \( ax + by \leq 0 \) is achieved - \( ax - by \in P \). Now from condition of (ii) in definition 2.1. - \( -ax, -by \in P \) and finally from condition of (iii) in definition 2.1. \( ax = by = 0 \), which implies \( x = y = 0 \).

Definition 2.2. Let \( (X, G) \) be a \( G \)-cone metric space, \( G \) is said to be continuous at \( x_0 \in E \) if for any sequence \( \{x_n\} \) in \( P \), \( x_n \rightarrow x_0 \) implies \( G(x_n, x_0) \rightarrow G(x_0, x_0) \). \( \varphi : P \rightarrow P \) is continuous if \( \varphi \) is continuous at all \( x_0 \in P \).

In the following it is suppose that \( E \) is a Banach space, \( P \) is a cone in \( E \) and \( \leq \) is partial ordering with respect to \( P \).

Definition 2.3. Let \( X \) be a nonempty set and suppose \( G : X \times X \times X \rightarrow E \) satisfies:

\((G_1)\) \( G(x,y,z) = 0 \) if \( x = y = z \).
\((G_2)\) \( 0 \leq G(x,y,z) \) whenever \( x \neq y \), \( \forall x,y \in X \).
\((G_3)\) \( G(x,y,z) \leq G(x,z,y) \) whenever \( y \neq z \).
\((G_4)\) \( G(x,y,z) = G(x,z,y) = G(y,x,z) \) \( \cdots \) (symmetry in all three variables).
\((G_5)\) \( G(x,y,z) \leq G(x,a,a) + G(a,y,z) \), \( \forall x,y,z,a \in X \).

Then \( G \) is called a generalized cone metric on \( X \), and \( (X,G) \) is called a generalized cone metric space or a \( G \)-cone metric space.

Corollary 2.1. if \( (X,G) \) be a \( G \)-cone metric space then
\( G(x,y,y) \leq G(x,z,z) + G(y,z,y) \), for all \( x,y,z \in X \).

Proof. It is sufficient be replaced \( z \) with \( y \) and \( a \) with \( z \) in \((G_5)\) from definition 2.3.

Definition 2.4. A \( G \)-cone metric space is said to be symmetric if:
\( G(x,y,y) = G(y,x,x) \), \( \forall x,y \in X \).

Following is example of non symmetric \( G \)-cone metric space.

Example 2.1([10]). Let \( X = \{a,b\}, E = \mathbb{R}^2, P = \{(x,y,z) \in E \mid x,y,z \leq 0\} \). Define \( G : X \times X \times X \rightarrow E \) by
\( G(a,a,a) = (0,0,0) = G(b,b,b) \).
\( G(a,b,b) = (0,1,1) = G(b,a,b) = G(b,b,a) \).
\( G(b,a,a) = (0,1,0) = G(a,b,a) = G(a,a,b) \).

Here \( X \) is non symmetric \( G \)-cone metric space since \( G(a,a,b) \neq G(a,b,b) \).

Definition 2.5. Let \( (X,G) \) be a \( G \)-cone metric space and \( \{x_n\} \) be a sequence in \( X \). It is said that \( \{x_n\} \) is
\( (1) \) a \( G \)- Cauchy sequence if, for every \( c \in E \), with \( 0 \ll c \), there is \( n_0 \in \mathbb{N} \) such that for \( n,m > n_0 \), \( G(x_m,x_n) \ll c \).
\( (2) \) a \( G \)-convergent sequence to \( x_0 \in X \) if, for every \( c \in E \), with \( 0 \ll c \), there is \( n_0 \in \mathbb{N} \) such that for \( n,m > n_0 \), \( G(x_m,x_n) \ll c \). Here \( x_0 \) is called the limit of the sequence \( \{x_n\} \).

A \( G \)-cone metric space \( (X,G) \) is said to be \( G \)-complete if every \( G \)-Cauchy sequence in \( X \) is \( G \)-convergent in \( X \). The following results are proved in [1].

Lemma 2.4. If \( (X,G) \) is a \( G \)-cone metric space, then the following are equivalent:
\( (i) \) \( \{x_n\} \) converges to \( x \).
\( (ii) \) \( G(x_n,x_n,x) \rightarrow 0 \) as \( n \rightarrow \infty \).
\( (iii) \) \( G(x_n,x) \rightarrow 0 \) as \( n \rightarrow \infty \).
\( (iv) \) \( G(x,x,x) \rightarrow 0 \) as \( n,m \rightarrow \infty \).

Lemma 2.5. Let \( \{x_n\} \) be a sequence in \( X \), if for every \( c \in E \), with \( 0 \ll c \), there is \( n_0 \in \mathbb{N} \) such that for \( n,m > n_0 \), \( G(x_m,x_n) \ll c \), then \( \{x_n\} \) is a \( G \)-Cauchy sequence in \( X \).

Proof. Suppose \( 0 \ll c \) is an arbitrarily element. According to the condition of lemma there is \( n_0 \in \mathbb{N} \) such that for \( n,m > n_0 \), \( G(x_{m},x_{n}) \ll \frac{c}{3} \) and \( G(x_{m},x_{n}) \ll \frac{c}{3} \). Using the property (G5) of Definition 2.3 will be obtained \( G(x_{m},x_{n}) \leq G(x_{m},x_{n}) + G(x_{n},x_{n}) \ll c \) and therefore, \( \{x_n\} \) is a \( G \)-Cauchy sequence in \( X \).
Lemma 2.6. Let $(X,G)$ be a $G$-cone metric space and let $(x_n)$ be a sequence in $X$. If there exists the number $k \in (0,1)$ such that
\begin{equation}
G(x_{n+2}, x_{n+1}) \leq k^2 G(x_{n+1}, x_n), \quad n=1,2,\ldots
\end{equation}
Then $(x_n)$ is a $G$-Cauchy sequence in $X$.

Proof. Using the condition (2.1) to be achieved
\begin{equation}
G(x_{n+2}, x_{n+1}) \leq k G(x_{n+1}, x_n) \leq \cdots \leq k^n G(x_1, x_0)
\end{equation}
So for $n > m$ according to Corollary 2.1
\begin{equation}
G(x_{n}, x_{m}) \leq G(x_{n}, x_{n+1}) + G(x_{n+1}, x_{n+2}) + \cdots + G(x_{m+1}, x_m) 
\leq (k^{n-m+1} + k^{n-m+2} + \cdots + k^m) G(x_1, x_0).
\end{equation}

Let $0 < \epsilon$ be given. Choose $\delta > 0$ such that $c + \frac{k}{\epsilon} < \delta$, where
\begin{equation}
N_\delta(0) = \{y \in X : \|y\| < \delta\},
\end{equation}
Also choose a natural number $n_0$ such that
\begin{equation}
\frac{k^n}{1+k} G(x_1, x_0) < \epsilon
\end{equation}
Thus $G(x_n, x_m) \leq \frac{k^n}{1+k} G(x_1, x_0) < \epsilon$ and hence, according to the lemma 2.5, $(x_n)$ is a $G$-Cauchy sequence in $X$.

Definition 3.1. Let $(X,G)$ be a $G$-cone metric space and let $T$ be an onto self mapping on $X$. Then $T$ is called an expansive mapping if there exist a constant $k > 1$ such that
\begin{equation}
G(Tx, Ty, Tz) \geq k G(x, y, z)
\end{equation}
for all $x, y, z \in X$.

Definition 3.2. Let $T$ and $S$ be self mappings of a set $X$. If $x$ is a coincidence point of $T$ and $S$ and $w$ is called a point of coincidence of $T$ and $S$. For any $x \in X$, the following holds:
\begin{equation}
T(x) = S(x).
\end{equation}

Definition 3.3. The mappings $T$ and $S$ are weakly compatible, if for every $x \in X$, the following holds:
\begin{equation}
T(S(x)) = S(T(x)).
\end{equation}

There are many theorems that prove existence of a fixed point for a mapping satisfies in expansive conditions. For example see next two theorems.

Theorem 2.1([10]). Let $(X,G)$ be a complete $G$-cone metric space. If there exist a constant $k > 1$ and an onto self mapping $T$ on $X$ satisfying
\begin{equation}
G(Tx, Ty, Tz) \geq k G(x, y, z)
\end{equation}
for all $x, y, z \in X$, then $T$ has a unique fixed point.

Theorem 2.2([10]). Let $(X,G)$ be a complete $G$-cone metric space and let $T : X \to X$ is an onto mapping satisfying
\begin{equation}
G(Tx, Ty, Tz) \geq k \cdot m
\end{equation}
for all $x, y, z \in X$, and constant $k > 1$, then $T$ has a unique fixed point.

3. Main results:
In this section will be presented some fixed point theorems for expansive mappings.

Theorem 3.1 Let $(X,G)$ be a complete symmetric $G$-cone metric space and let $T : X \to X$ be an onto mapping satisfying
\begin{equation}
G(Tx, Ty, Tz) \geq a G(x, y, z) + b G(Tx, Tx) + c G(Ty, Ty)
\end{equation}
for all $x, y, z \in X$, and constant $k > 1$, then $T$ has a unique fixed point.

Proof. Since $T : X \to X$ is onto then $T$ is injective and invertible. Suppose that $H$ be the inverse mapping of $T$. Let $x_0 \in X$, so there exists $x_1 = H(x_0)$ such that $x_1 = H(x_0)$. Continuing in this way we can obtain a sequence $(x_n)$ in $X$ where $x_n = T(x_{n-1})$.

If $x_{n+1} \neq x_n$ for any $n = 1,2,\ldots$ then $X_n$ is a fixed point of $T$ and proof is complete. So suppose that $x_{n+1} = x_n$ for all $n = 1,2,\ldots$

It follows that from condition (3.1)
\begin{equation}
G(x_{n+1}, x_n) = G(T^{-1}x_{n-1}, T^{-1}x_n, T^{-1}x_n) \geq a G(T^{-1}x_{n-1}, T^{-1}x_n, T^{-1}x_n) + b G(T^{-1}x_{n-1}, T^{-1}x_n, T^{-1}x_n) + c G(T^{-1}x_{n-1}, T^{-1}x_n, T^{-1}x_n)
\end{equation}
Since $G$-cone metric space is assumed symmetric then from (3.2) is obtained
If \((a+c)=0\), then \(b > 0\) or \(1-b<0\) and hence The inequality (3.3) implies that
\[
(1-b)g(x_{n-1},x_n) \geq 0.
\]
also according the cone property, \((b-1)g(x_{n-1},x_n) \geq 0\) and thus \(g(x_{n-1},x_n)=0\) or
\[
x_{n-1} = x_n, \quad \text{that is contradiction. Hence} \quad (1-b)<0 \quad \text{and} \quad 1-b>0. \quad \text{Therefore}
\]
\[
G(x_{n-1},x_n) \geq \frac{1}{b} g(x_{n-1},x_n).
\]
Since \(\frac{1}{b} \leq 1\), by lemma 2.6 \(\{x_n\}\) is a G-Cauchy sequence in \(X\). Because \((X,G)\) is complete and therefore the sequence \(\{x_n\}\) is converges to a point \(w \in X\). Since \(T\) is onto mapping so there exists \(u \in X\), such that \(w = Tu\). Now from condition (3.1),
\[
G(x_n,w,u) \geq G(Tx_{n+1},Tu,u) \geq aG(x_{n+1},u,u) + bG(x_n,u,u) + cG(u,u,u).
\]
Which implies that as \(n \to \infty\)
\[
G(w,w,w) \geq aG(w,w,u) + bG(w,w,w) + cG(w,w,w) \quad \text{or} \quad 0 \geq (a+c)G(w,w,u).
\]
Hence \(G(w,w,u) = 0\) and therefore \(Tu = w\). This gives that \(w\) is a fixed point of \(T\) and hence proof is complete. \(\square\)

Theorem 3.2. Let \((X,G)\) be a complete G-cone metric space and let \(T: X \to X\) be a onto mapping satisfying
\[
G(Tx,Ty,Tz) \geq k(G(x,y,z) + G(Ty,Tz,Tx))
\]
for all \(x,y,z \in X\), and constant \(k \geq \frac{1}{2}\). Then \(T\) has a fixed point.

Proof. Similar to the proof of Theorem 2.1, we can obtain a sequence \(\{x_n\}\) such that \(x_{n+1} = Tx_n\). If for a positive integer \(n\), \(x_{n-1} = x_n\) then \(x_n\) is a fixed point of \(T\) and proof is complete. So suppose that \(x_{n-1} \neq x_n\) for all \(n = 1, 2, \ldots\)
It follows that from condition (3.5)
\[
G(x_{n+1},x_n,x_{n-1}) \geq k(G(x_{n+1},x_n,Tx_n) + G(x_n,x_{n+1},Tx_{n+1})).
\]
Taking \(n \to \infty\), in above inequality
\[
G(x_{n+1},x_n,x_{n-1}) \rightarrow 0 \quad \text{as} \quad n \to \infty.
\]
Thus by lemma 2.6 \(\{x_n\}\) is a G-Cauchy sequence in \(X\) and hence is G-converges to a point \(w \in X\). Since \(T\) is onto mapping so there exists \(u \in X\), such that \(w = Tu\). Now from condition (3.5),
\[
G(x_n,w,u) = G(Tx_{n+1},Tu,u) \geq k(G(x_{n+1},u,u) + G(u,u,u)).
\]
Therefore, \(w\) is a fixed point of \(T\). \(\square\)

Now, is presented a common fixed point theorem of two weakly compatible mappings in G-cone metric spaces.

Theorem 3.3. Let \((X,G)\) be a G-cone metric space. Let \(S\) and \(T\) be weakly compatible self-mapping of \(X\). Suppose that there exists \(k \geq 1\) such that
\[
G(Sx,Sy,Sz) \geq kG(Tx,Ty,Tz)
\]
for all \(x,y,z \in X\). If one of the subspaces \(T(X)\) or \(S(X)\) is complete, then \(S\) and \(T\) have a unique common fixed point in \(X\).

Proof. Let \(x_0 \in X\). Since \(T(X) \subseteq s(X)\), choose \(x_1 \in X\) such that \(Sx_1 = Tx_0 = z_0\). In general for every positive integer \(n\), choose \(x_{n+1} \in X\) such that \(Sx_{n+1} = Tx_n = z_n\). Then from (3.7),
\[
G(z_{n+1},z_n) = G(Tx_{n+1},Tx_n, Tx_n) \geq \frac{1}{k} G(Sx_{n+1},Sx_n, Sx_n) = \frac{1}{k} G(z_{n+1},z_n, z_{n-1}).
\]
Thus by lemma 2.6, \(\{z_n\}\) is a G-Cauchy sequence in \(\tau(X) \subseteq s(X)\) and hence is G-converges to a point \(u \in X\). (Because Under the assumption \(T(X)\) or \(S(X)\) is complete.)

Thus, \(\lim_{n \to \infty} z_n = \lim_{n \to \infty} Tz_n = \lim_{n \to \infty} z_n = u\). Since \(T(X)\) or \(S(X)\) is complete and \(\tau(X) \subseteq s(X)\), there exists a point \(w \in X\) such that \(Sw = u\). Now from (3.7)
$$G(Tw,Tx_{n},Tx_{n}) \leq \frac{1}{k} G(Sw,Sx_{n},Sw_{n}) \quad (3.8)$$

Taking $n \rightarrow \infty$ in (3.8) therefore, $G(Tw,Tw_{n}) \leq \frac{1}{k} G(Sw,Sw_{n}) = 0$ which implies that $Tw=Tu$ and hence, $Tw=Sw=Tu$. Since $S$ and $T$ are weakly compatible, therefore, $STw=TSw$ or $Su=Tu$. Now it can be shown that $u$ is a fixed point of $S$ and $T$.

For this purpose from (3.7)

$$G(Su,Sx_{n},Sw_{n}) \leq \frac{1}{k} G(Tu,Tx_{n},Tu_{n}) \quad (3.9)$$

Taking $n \rightarrow \infty$ in (3.8) therefore, $G(Su,Sw_{n}) \leq \frac{1}{k} G(Tu,Sw_{n}) = 0$ which implies that $Su=Sw=Tu$. Thus $Su=Tu=U$. To prove uniqueness suppose that, $y$ is also another common fixed point of $S$ and $T$ ($Sy=Ty=y$). Now from (3.7)

$$G(Su,Sw,Sw) \geq kG(Ty,Ty,Sw) \quad \text{or} \quad G(u,y,y) \geq kG(u,y,y)$$

which implies that $G(u,y,y) = 0$ and hence, $u=y$. This completes the proof.

Corollary 3.1. Let $(X,G)$ be a complete $G$-cone metric space and $S:X \rightarrow X$ be a surjection and $T:X \rightarrow X$ be an injective. If $S$ and $T$ are commutative, and there is constant $k \geq 1$ such that

$$G(XG,SY,SY) \geq kG(XG,TY,TY) \quad \text{for all} \ x,y \in X.$$

Then $S$ and $T$ have a unique common fixed point in $X$.

Proof. Clearly $S$ and $T$ are weakly compatible and since $S$ is surjection then $S(X) = X$ and therefore $S(X)$ is $G$-complete. Also $T(X) \subseteq S(X)$. Thus all conditions of theorem 3.3 are satisfied and hence $S$ and $T$ have a unique common fixed point in $X$.

Now, an example is provided to illustrate the corollary 3.1.

Example 3.1. Let $X = [1, \infty)$, $E = \mathbb{R}$ and $p = \{x \in \mathbb{R} | x \geq 0\}$ be a cone in $E$.

Define $G: X \times X \times X \rightarrow E$ by

$$G(x,y,z) = |x-y|+|y-z|+|z-x|$$

for all $x,y,z \in X$. Then $X$ is a complete $G$-cone metric space. Define $TS:X \rightarrow X$ by $Tx=4x-3$ and $Sy=5x-4$ for all $x \in X$. Then following conditions is satisfied.

(1). $S$ and $T$ are weakly compatible.

(2). $S$ is surjection (range of $S$ is $X = [1, \infty)$) and $T$ is injective and $T(X) \subseteq S(X)$.

(3). $S(X) = X$ is complete.

(4). $G(Sx,Sy,Sy) = 2|Sx-Sy| = 10|x-y|$ and $G(Tx,Ty,Ty) = 2|Tx-Ty| = 8|x-y|$.

Let $k = \frac{5}{4}$. For all $x,y \in X$ the following is satisfied.

$$G(Sx,Sy,Sy) \geq kG(Tx,Ty,Ty)$$

Thus according to corollary 3.1, $S$ and $T$ have a unique common fixed point in $X$.

$u = 1$, is unique common fixed point of $S$ and $T$. ($T(1) = S(1) = 1$).

REFERENCES


