Stability of Continuous G-frame in Hilbert C*-modules

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ABSTRACT

In this paper, we give a generalization of stability for g-frame in Hilbert C*-modules that was introduced by Rashidi and Nazari. This generalization is a natural generalization of stability for continuous and discrete g-frames and frame in Hilbert space too. First, we give some characterization of continuous g-frames and g-Riesz bases in Hilbert C*-modules. Then, we develop the basic stability theory for continuous g-frames and g-Riesz bases in Hilbert C*-module.

INTRODUCTION

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [10] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies et al. [9] and popularized from then on. The theory of frames plays an important role in application [3,13].

Let \( H \) be a Hilbert space, and \( I \) a set which is finite or countable. A system \( \{ f_i \}_{i \in I} \subset H \) is called a frame for \( H \) if there exist the constants \( A, B > 0 \) such that

\[
A \| f \|_2^2 = \sum_{i \in I} |(f, f_i)^* | \leq B \| f \|_2^2
\]

For all \( f \in H \). The constants \( A \) and \( B \) are called frame bounds. If \( A = B = 1 \) it is called a Parseval frame. We refer the readers to references [7,8] for more details on frames.

In [25] Sun introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

Let \( U \) and \( V \) be two Hilbert spaces, and \( \{ V_i : i \in I \} \) is a sequence of subspaces of \( V \), where \( I \) is a subset of \( \mathbb{Z} \). \( L(U, V_i) \) is the collection of all bounded linear operators from \( U \) into \( V_i \). We call a sequence \( \{ A_i \in L(U, V_i) : i \in I \} \) a generalized frame, or simply a g-frame, for \( U \) with respect to \( \{ V_i : i \in I \} \) if there are two positive constants \( A \) and \( B \) such that

\[
A \| f \|_U^2 = \sum_{i \in I} |(A_i f, f_i)^* | \leq B \| f \|_U^2
\]

For all \( f \in U \). The constants \( A \) and \( B \) are called g-frame bounds. If \( A = B = 1 \) it is called a Parseval g-frame.

On the other hand, the concept of frames especially the g-frames was introduced in Hilbert C*-modules, and some of their properties were investigated in [14-17,19,20,23,24,29]. Frank and Larson [14] defined the standard frames in Hilbert C*-modules and got a series of results for standard frames in finitely or countably generated Hilbert C*-modules over unital C*-algebras. As for Hilbert C*-module, it is a generalization of Hilbert spaces in that it allows

The inner product to take values in a C*-algebra rather than the field of complex numbers.

There are many differences between Hilbert C*-modules and Hilbert spaces. For example, we know that any closed subspace in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C*-module. And we cannot get the analogue of the Riesz representation theorem of continuous functional C*-modules generally. Thus it is more difficult to make a discussion of the theory of Hilbert C*-modules than that of Hilbert C* modules.

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spaces in general. We refer the readers to references [21, 28] for more details on Hilbert C*-modules. In [20, 29], the authors made a discussion of some properties of g-frame in Hilbert C*-module in some aspects.

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by Kaiser [18] and independently by Ali et al. [1]. These frames are known as continuous frames. Let \( H \) be a Hilbert space, and let \((M, S, \mu)\) be a measure space. A continuous frame in \( H \) indexed by \( M \) is a family \( h = \{h_m \in H : m \in M\} \) such that:

1. For any \( f \in H \), the function \( f : M \to \mathbb{C} \) defined by \( f_m = \langle h_m, f \rangle \) is measurable;
2. There is a pair of constants \( 0 < A, B \) such that, for any \( f \in H \),
   \[
   A \|f\|_A^2 \leq \sum_{m \in M} |\langle f, h_m \rangle|^2 \leq B \|f\|_B^2
   \]

If \( M = \mathbb{N} \) and \( \mu \) is the counting measure, the continuous frame is a frame.

Rashidi and Nazari defined continuous g-frames in Hilbert C*-modules [23]. This definition is a generalization of all previous definitions.

Let \( H \) be a Hilbert space with Riesz basis \( \{f_i\}_{i=1}^\infty \) and let \( \{g_i\}_{i=1}^\infty \) be a sequence of vectors in \( H \). If there exists a constant \( \lambda \in \mathbb{C}^* \) such that

\[
\left\| \sum_{i=1}^n c_i (f_i - g_i) \right\| \leq \lambda \left\| \sum_{i=1}^n c_i f_i \right\|
\]

for all finite sequence \( \{c_i\} \) of scalars, then \( \{g_i\}_{i=1}^\infty \) is also a Riesz basis for \( H \). This result is the well-known classical Paley-Wiener Theorem on perturbation of Riesz basis in Hilbert space [22]. Note the later condition implies that there exists a bounded invertible operator \( T \) such that \( T f_i = g_i \) [30]. This observation enables one to investigate the perturbation of bases and frames from the operator point of view [2, 4]. In the last decade, several authors have generalized the Paley-Wiener perturbation theorem to the perturbation of frames in Hilbert spaces [2, 4-6]. The following is a fundamental result due to Casazza and Christensen [4].

Theorem 1.1. Let \( \{f_i\}_{i=1}^\infty \) be a frame in a Hilbert space \( H \) with frame bounds \( A \) and \( B \). Assume that \( \{g_i\}_{i=1}^\infty \) is a sequence of vectors in \( H \) and there exist \( \lambda, \nu, \mu \geq 0 \) such that \( \max \left\{ \lambda, \frac{\nu}{\lambda}, \frac{\mu}{\lambda} \right\} < 1 \). Suppose one of the following conditions holds for any finite sequence \( \{\epsilon_i\} \) and every \( f \in H \). Then \( \{g_i\}_{i=1}^\infty \) is also a frame in \( H \).

\[
\left( \sum_{i=1}^n |\langle f, f_i - g_i \rangle|^2 \right)^{1/2} \leq \lambda \left( \sum_{i=1}^n |\langle f, f_i \rangle|^2 \right)^{1/2} + \lambda \left( \sum_{i=1}^n |\langle f_i, g_i \rangle|^2 \right)^{1/2} + \nu \|f\|_A \left\| \sum_{i=1}^n \epsilon_i f_i \right\| + \mu \|f\|_B \left\| \sum_{i=1}^n \epsilon_i g_i \right\| + \nu \left( \sum_{i=1}^n |\epsilon_i|^2 \right)^{1/2} + \mu \left( \sum_{i=1}^n |\epsilon_i|^2 \right)^{1/2}
\]

The paper is organized as follows. In Sections 2 we recall the basic definitions and some notations about continuous g-frames in Hilbert C*-module; we also give some basic properties of g-frames which we will use in the later sections. In Section 3, we give some results of stability for continuous g-frames in Hilbert C*-modules.

2. Preliminaries:

In the following we review some definitions and basic properties of Hilbert C*-modules and g-frames in Hilbert C*-module; we first introduce the definition of Hilbert C*-modules.

Definition 2.1. Let \( A \) be a C*-algebra with involution *\(^a\). An inner product \( A\)-module (pre-Hilbert \( A\)-module) is a complex linear space \( H \) which is a left \( A\)-module with map \( <, \cdot : H \times H \to A \) which satisfies the following properties:

1. \( \alpha f + \beta g, h > = \alpha < f, h > + \beta < g, h > \) for all \( f, g, h \in H \) and \( \alpha, \beta \in \mathbb{C} \);
2. \( \alpha f, g > = \alpha < f, g > \) for all \( f, g \in H \) and \( \alpha \in A \);
3. \( < f, g > \geq < g, f >^* \) for all \( f, g \in H \);
4. \( < f, f > \geq 0 \) for all \( f \in H \) and \( < f, f > = 0 \) if and only if \( f = 0 \).

For \( f \in H \), we define a norm on \( H \) by \( \|f\|_A = \|< f, f > \|^{1/2} \). Let \( H \) be complete with this norm, it is called a Hilbert C*-module over \( A \) or a Hilbert \( A \)-module.

An element \( a \) of a C*-algebra \( A \) is positive if \( a^* = a \) and spectrum of \( a \) is a subset of positive real number. We write \( a \geq 0 \) to mean that \( a \) is positive. It is easy to see that \( < f, f > \geq 0 \) for every \( f \in H \), hence we define \( |f| = < f, f >^{1/2} \).

Frank and Larson [14] defined the standard frames in Hilbert C*-modules. If \( H \) be a Hilbert C*-module, and \( f \) a set which is finite or countable. A system \( \{f_i\}_{i=1}^\infty \subseteq H \) is called a frame for \( H \) if there exist the constants \( A, B > 0 \) such that

\[
A \|f\|_A \leq \sum_{i=1}^\infty \langle f, f_i \rangle \|f_i\| \leq B \|f\|_B
\]

for all \( f \in H \). The constants \( A \) and \( B \) are called frame bounds.
A. Khosravi and B. Khosravi [20] defined g-frame in Hilbert C*-module. Let \( U \) and \( V \) be two Hilbert C*-module, and \( \{V_i : i \in I\} \) is a sequence of subspaces of \( V \), where \( I \) is a subset of \( \mathbb{Z} \) and \( \text{End}_A^*(U,V_i) \) is the collection of all adjointable \( A \)-linear maps from \( U \) into \( V_i \) that is, \( \langle Tf, g \rangle = \langle f, \Gamma_T^* g \rangle \) for all \( f, g \in H \) and \( T \in \text{End}_A^*(U,V_i) \). We call a sequence \( \{A_i \in \text{End}_A^*(U,V_i) : i \in I\} \) a generalized frame, or simply a g-frame, for Hilbert C*-module \( U \) with respect to \( \{V_i : i \in I\} \) if there are two positive constants \( A \) and \( B \) such that
\[
A(f,f) \leq \sum_{i \in I} \langle A_i f, A_i f \rangle \leq B(f,f)
\]
for all \( f \in U \). The constants \( A \) and \( B \) are called g-frame bounds. If \( A = B \) we call this g-frame a tight g-frame, and if \( A = B = 1 \) it is called a Parseval g-frame.

Rashidi and Nazari defined continuous g-frames in Hilbert C*-module [23]. Let \((M,S\mu)\) be a measure space, let \( U \) and \( V \) be two Hilbert C*-modules, \( \{V_m : m \in M\} \) is a sequence of subspaces of \( V \), and \( \text{End}_A^*(U,V_m) \) is the collection of all adjointable \( A \)-linear maps from \( U \) into \( V_m \).

Definition 2.2. We call a net \( \{A_m \in \text{End}_A^*(U,V_m) : m \in M\} \) a continuous generalized frame, or simply a continuous g-frame, for Hilbert C*-module \( U \) with respect to \( \{V_m : m \in M\} \) if
1. for any \( f \in U \), the function \( f : M \to V_m \) defined by \( f_m = A_m f \) is measurable;
2. there is a pair of constants \( 0 < A, B \) such that, for any \( f \in U \),
\[
A(f,f) \leq \int_M \langle A_m f, A_m f \rangle d\mu(m) \leq B(f,f)
\]

The constants \( A \) and \( B \) are called continuous g-frame bounds. If \( A = B \) we call this continuous g-frame a continuous tight g-frame, and if \( A = B = 1 \) it is called a continuous Parseval g-frame. If only the right-hand inequality is satisfied, we call \( \{A_m : m \in M\} \) the continuous \( g \)-Bessel for \( U \) with respect to \( \{V_m : m \in M\} \) with Bessel bound \( B \).

If \( M = \mathbb{N} \) and \( \mu \) is the counting measure, the continuous g-frame for \( U \) with respect to \( \{V_m : m \in M\} \) is a g-frame for \( U \) with respect to \( \{V_m : m \in M\} \).

Let \( \{A_m : m \in M\} \) be a continuous g-frame for \( U \) with respect to \( \{V_m : m \in M\} \). Define the continuous g-frame operator \( S \) on \( U \) by
\[
Sf = \int_M A_m f d\mu(m).
\]
Proposition 2.3. The frame operator \( S \) is a bounded, positive, selfadjoint, and invertible.

Theorem 2.4. Let \( \Lambda_m \in \text{End}_A^*(U,V_m) \) for any \( m \in M \). Then \( \{\Lambda_m : m \in M\} \) is a continuous g-frame for \( U \) with respect to \( \{V_m : m \in M\} \) if and only if there exist constants \( A, B > 0 \) such that for any \( f \in U \),
\[
A\|f\|^2 \leq \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq B\|f\|^2
\]
We define
\[
\Theta_{m \in M} V_m = \left\{ g = \{g_m : g_m \in V_m : \int_M \langle g_m, g_m \rangle d\mu(m) < \infty\} \right\}.
\]
For any \( f = \{f_m : m \in M\} \) and \( g = \{g_m : m \in M\} \), if the \( A \)-valued inner product is defined by
\[
< f, g > = \int_M < f_m, g_m > d\mu(m),
\]
the norm is defined by \( \|f\| = \|< f, f > \|^{1/2} \), then \( \Theta_{m \in M} V_m \) is a Hilbert \( A \)-module.

Let \( \{\Lambda_m \in \text{End}_A^*(U,V_m) : m \in M\} \) be a continuous g-frame for \( U \) with respect to \( \{V_m : m \in M\} \), we define synthesis operator \( T : \Theta_{m \in M} V_m \to U \) by
\[
T(g) = \int_M \Lambda_m^* g_m d\mu(m)
\]
for all \( g = \{g_m : m \in M\} \). The analysis operator is defined for map \( F : U \to \Theta_{m \in M} V_m \) by \( F(f) = \{\Lambda_m : m \in M\} \) for any \( f \in U \).

Theorem 2.4. A net \( \{\Lambda_m \in \text{End}_A^*(U,V_m) : m \in M\} \) is a continuous g-frame for \( U \) with respect to \( \{V_m : m \in M\} \) if and only if synthesis operator \( T \) is well defined and surjective.

Definition 2.5. A continuous g-frame \( \{\Lambda_m \in \text{End}_A^*(U,V_m) : m \in M\} \) in Hilbert C*-module \( U \) with respect to \( \{V_m : m \in M\} \) is said to be a continuous g-Riesz basis if it satisfies that
1. \( \Lambda_m = 0 \) for any \( m \in M \);
2. if \( \int_M \Lambda_m^* g_m d\mu(m) \) then every summand \( \Lambda_m^* g_m \) is equal to zero, where \( \{g_m : m \in M\} \in \Theta_{m \in M} V_m \) and \( K \) is a measurable subset of \( M \).

Now we give some characterizations of continuous g-Riesz bases and continuous g-frames in Hilbert C*-modules.

Let \( P_m \) be the projection on \( \Theta_{m \in M} V_m \) that is, \( P_m : \Theta_{m \in M} V_m \to \Theta_{m \in M} V_m \) is defined by \( P_m f = U_j \), for \( f = \{f_m\}_{m \in M} \in \Theta_{m \in M} V_m \), and \( U_j = f_m \) when \( j = m \) and \( U_j = 0 \) when \( j \neq m \).
Theorem 2.6. Let \( \{\Lambda_m \in \text{End}^2(A(U, V_m)) : m \in M\} \) be a continuous g-frame in Hilbert A-module U with respect to \( \{V_m : m \in M\} \), then \( \{\Lambda_m \in \text{End}^2(A(U, V_m)) : m \in M\} \) is a continuous g-Riesz basis if and only if

\[ \Lambda_m = \cdot \quad \text{and} \quad P_m(\text{Rang} T) \subseteq \text{Rang} T \quad \text{for all} \quad m \in M, \]

where T is the analysis operator of \( \{\Lambda_m \in \text{End}^2(A(U, V_m)) : m \in M\} \).

Proof. Let \( \{\Lambda_m \in \text{End}^2(A(U, V_m)) : m \in M\} \) be a continuous g-frame. Suppose that

\[ g = \{g_m\}_{m \in M} \in (\text{Rang} T)^* \]

\[ \langle f, \Lambda_m g \rangle = = \langle f, T^* g \rangle = = \]

\[ f, \int_M \Lambda_m^* g d\mu(m) \]

for all \( f \in U \). This implies that \( \int_M \Lambda_m^* g d\mu(m) = \cdot \) and therefore \( \Lambda_m^* g_m = \cdot \) for all \( m \in M \). Hence \( \langle f, \Lambda_m f \rangle = = \langle \Lambda_m f, g \rangle = = \) and \( g = \{g_m\}_{m \in M} \in [\text{Rang} T]^* \). Therefore \( (\text{Rang} T)^* \subseteq [\text{Rang} T]^* \).

Now, suppose that \( P_m(\text{Rang} T) \subseteq \text{Rang} T \) for each \( m \in M \). Let \( \int_M \Lambda_m^* g d\mu(m) = \cdot \) where \( \{g_m\}_{m \in M} \in \text{End}^2(A(U, V_m)) \). Fix \( m \in M \), then \( P_m T f \in \text{Rang} T \), so there exists \( h_m \in U \) such that \( \Lambda_m^* g_m = \cdot \).

Now, we generalize proposition 2.1 of [24] on characterization of g-frames and g-Riesz bases. The proof is similar and we leave it to interested readers.

Proposition 2.7. Let \( f \) be an operator sequence in Hilbert C*-module U with respect to \( \{V_m : m \in M\} \). If the operator F satisfies

\[ \text{span}[\Lambda_m^* f_m : f_m \in V_m, m \in M] = U \]

and operator F is bounded and satisfies

\[ \sqrt{\lambda} \|f\| \leq \|F(f)\| \leq \sqrt{\mu} \|f\|, \quad f = \{f_m\}_{m \in M} \in (\text{ker} F)^* \].

Furthermore, \( \{\Lambda_m^* f_m : f_m \in V_m, m \in M\} \) is a g-Riesz basis with unique dual g-frame if and only if

\[ \text{span}[\Lambda_m^* f_m : f_m \in V_m, m \in M] = U \]

and there exist \( \lambda, \mu > 0 \) such that

\[ \sqrt{\lambda} \|f\| \leq \|F(f)\| \leq \sqrt{\mu} \|f\|, \quad f = \{f_m\}_{m \in M} \in \text{End}^2(A(U, V_m)) \].

3. Stability of Continuous G-Frames In Hilbert C*-Modules:

The stability of frames is important in practice and is studied widely by many authors, see [2, 4-6, 11, 12, 17, 24, 26, 27]. The main result of this paper is to extend the Casazza-Christensen perturbation theorem to continuous g-frames in Hilbert C*-modules.

We need the following lemma due to Casazza and Christensen [4]. It is a generalization of the classical result that an operator U on a Banach space is invertible if \( \|I - U\| < 1 \).

Lemma 3.1. Let X be a Banach space and \( U : X \rightarrow X \) be a linear operator. Assume that there exist constants \( \lambda, \lambda \in (0,1) \) such that

\[ \|UX - x\| \leq \lambda \|x\| + \lambda \|UX\|, \quad x \in X. \]

Then U is bounded and invertible with

\[ \|U\| \leq \frac{1 + \lambda}{1 - \lambda}, \quad \|U^{-1}\| \leq \frac{1 - \lambda}{1 + \lambda}. \]

Theorem 3.2. Let \( \{\lambda_m \in \text{End}^2(A(U, V_m)) : m \in M\} \) be a continuous g-frame in Hilbert C*-module U with respect to \( \{V_m : m \in M\} \) with continuous g-frame bounds A and B. Suppose that \( \{\Gamma_m : m \in M\} \) is a net of operators of U to \( V_m \) and that there exist \( \lambda, \lambda_r, \mu > 0 \) such that \( \max \{\lambda, + \frac{\mu}{\sqrt{A}}, \lambda_r\} < 1 \). Then \( \{\Gamma_m : m \in M\} \) is also a continuous g-frame for U with respect to \( \{V_m : m \in M\} \) with continuous g-frame bounds

\[ \left( \frac{1 - \lambda \lambda_r \sqrt{A} - \mu}{1 + \lambda \lambda_r \sqrt{B} + \mu} \right), \quad \left( \frac{1 + \lambda \lambda_r \sqrt{B} + \mu}{1 - \lambda \lambda_r \sqrt{A} - \mu} \right). \]

If one of the following condition is fulfilled for any sequence \( \{f_k : k \in K\} \) that K is a subset of finite measure M and all \( f \in U \).
\[
\begin{align*}
&\left\| \int_M \langle (\Lambda_m - \Gamma_n)f, (\Lambda_m - \Gamma_n)f \rangle \, d\mu(m) \right\|_1^2 \\
\leq & \lambda \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle \, d\mu(m) \right\|_1^2 \\
+ & \lambda \left\| \int_M \langle \Gamma_n f, \Gamma_n f \rangle \, d\mu(m) \right\|_1^2 + \mu \|f\|_1^2 \\
& \left( \int_M \langle \Lambda_m f, \Lambda_m f \rangle \, d\mu(m) \right) \|f\|_{L^2}^2,
\end{align*}
\]

An important special case of Theorem 3.1 is as follows.

Corollary 3.3. Let \( \{\varphi_m \in \mathcal{H} \}_m \) be a continuous g-frame in Hilbert \( C^* \)-module \( U \) with respect to \( \{V_m : m \in M \} \) with continuous g-frame bounds \( A, B \) and let \( \{\Gamma_m : m \in M \} \) is a net of operators of \( U \) to \( V_m \). If there exists a constant \( R < A \) such that

\[
\left\| \int_M \langle (\Lambda_m - \Gamma_m)f, (\Lambda_m - \Gamma_m)f \rangle \, d\mu(m) \right\|_1^2 \leq \lambda \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle \, d\mu(m) \right\|_1^2 + \mu \|f\|_1^2 + \lambda \left( \int_M \langle \Lambda_m f, \Lambda_m f \rangle \, d\mu(m) \right) \|f\|_{L^2}^2
\]

for every \( f \in U \), then \( \{\Gamma_m : m \in M \} \) is a continuous g-frame for \( U \) with respect to \( \{V_m : m \in M \} \) with bounds

\[
A \left( 1 - \frac{R}{\sqrt{A}} \right), \quad B \left( 1 + \frac{R}{\sqrt{B}} \right)
\]

Proof. The condition in Corollary 3.1 corresponds to the condition in Theorem 3.1 with \( \lambda_1 = \lambda_2 = \cdot \) and \( \mu = \sqrt{R} \).

REFERENCES


